81. *Mordell-Weil Lattices and Galois Representation. II*

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This is a continuation of the previous note [0] and we use the same notation and the references given there. Detailed accounts will be published elsewhere.

2. Rational elliptic surfaces. From now on, we suppose that S is a *rational* elliptic surface with a section O; then $C=\boldsymbol{P}^1$, $K=k(t)$ (t: a variable over k). The assumption (*) on the non-smoothness of $f: S\to C$ is automatically satisfied. For such a surface S, the arithmetic genus $\chi=1$ and every section of f is an exceptional curve of the first kind on S, and conversely. The Néron-Severi group $NS(S)$ is a unimodular lattice of rank $\rho=10$, and (1.8) implies $r+\sum_{v\in R}(m_v-1)=8$, and hence $r\leq 8$.

For $r=8$, 7 or 6, the structure of the Mordell-Weil lattice $E(K)$ is completely described by the following theorem:

Theorem 2.1. (i) *If f has no reducible fibres ($R=\phi$), then $r=8$ and* $E(K)=E(K)^0\simeq E_8$.

(ii) *If there is only one reducible fibre $f^{-1}(v)$ and $m_v=2$, then $r=7$ and*

$$\begin{array}{ccc} E(K) & \simeq & E_7^* \\ \cup & & \cup \\ E(K)^0 & \simeq & E_7 \end{array} \qquad \textit{index } 2.$$

(iii) *If there is only one reducible fibre $f^{-1}(v)$ and $m_v=3$, then $r=6$ and*

$$\begin{array}{ccc} E(K) & \simeq & E_6^* \\ \cup & & \cup \\ E(K)^0 & \simeq & E_6 \end{array} \qquad \textit{index } 3.$$

(iv) *If there are only two reducible fibres $R=\{v, v'\}$ with $m_v=m_{v'}=2$, then $r=6$ and*

$$\begin{array}{ccc} E(K) & \simeq & D_6^* \\ \cup & & \cup \\ E(K)^0 & \simeq & D_6 \end{array} \qquad \textit{index } 4.$$

In the above, E_8, E_7, E_6 and D_6 stand for the root lattices of designated type and $*$ denotes their dual lattices (cf. [1, Ch. 6], [2, Ch. 4]). In particular, E_8 is a positive-definite even unimodular lattice, which is unique up to isomorphism. Of course, we could continue the above result to lower r; for instance, if $\sharp R=1$ and $m_v=4$ or 5, then $E(K)\simeq D_5^*$ or A_4^*, and so on.

Now, for any $P\in E(K)$, $P\neq O$, we have by (1.12):

$$(2.1) \qquad \langle P, P\rangle = 2+2(PO)-\begin{cases} 0 & \text{case (i) or } P\in E(K)^0 \\ 1/2 & \text{case (ii)}, \quad P\in E(K)-E(K)^0 \\ 2/3 & \text{case (iii)}, \quad \quad " \\ 1 \text{ or } 1/2 & \text{case (iv)}, \quad \quad " \end{cases}$$

This shows, in particular, that $E(K)$ is torsion-free in these cases. In general, we call $P \in E(K)$ a *minimal point* or *minimal section* if $\langle P, P \rangle$ has the smallest positive value in the Mordell-Weil lattice.

On the other hand, we know the following table ([1], [2] or [6, Ch. 4]):

Table

(2.2)

	E_8	E_7	E_7^*	E_6	E_6^*	D_6	D_6^*
determinant	1	2	1/2	3	1/3	4	1/4
minimal norm	2	2	3/2	2	4/3	2	1 (3/2)
♯min. vectors	240	126	56	72	54	60	12 (64)
Automorphisms	$W(E_8)$	$W(E_7)$		$W(E_6)\{\pm 1\}$		$W(D_6)\{\varepsilon\}$	
♯Aut	$2^{14}3^5 5^2 7$	$2^{10}3^4 5\cdot 7$		$2^7 3^4 5\cdot 2$		$2^5\cdot 6!\cdot 2$	

In the table, $W(E_8)$, etc. denote the Weyl groups.

Theorem 2.2. *The number of the minimal sections of the Mordell-Weil lattice* $E(K)$ *is* 240 *if* $r=8$, 56 *if* $r=7$, 54 *or* 12 *if* $r=6$. *They contain a minimal set of generators of* $E(K)$ *in case* (i), (ii) *or* (iii), *while the next minimal sections of norm* 3/2 *are also needed in case* (iv).

Corollary 2.3. *The Mordell-Weil group of a rational elliptic surface is generated by the sections* P *with* $\langle P, P \rangle \leq 2$, *at least if* $r \geq 6$.

Question 2.4. *Is the Mordell-Weil group of any elliptic surface generated by the sections* P *with* $\langle P, P \rangle \leq 2\chi$, χ *being the arithmetic genus of the surface?*

As for the torsion, we have:

Proposition 2.5. *The order of* $E(K)_{\text{tor}}$ *for a rational elliptic surface is one of the following values*: 1, 2, 3, 4, 5, 6, 8 *or* 9. *Conversely, each of these orders can occur.*

3. Weierstrass form. Let us make the results in § 2 more explicit by writing the equation of the elliptic curve E over $K=k(t)$ in the Weierstrass form. For simplicity, we assume char $(k) \neq 2, 3$. Suppose that the minimal Weierstrass equation of E over K is given by

(3.1) $$y^2 = x^3 + p(t)x + q(t), \qquad p(t),\ q(t) \in k[t].$$

Then the Kodaira-Néron model $f: S \to \boldsymbol{P}^1$ is a rational elliptic surface if and only if $\deg p(t) \leq 4$, $\deg q(t) \leq 6$ and $\Delta = 4p(t)^3 + 27q(t)^2$ is not a constant in k.

Lemma 3.1. *Let* $P=(x, y) \in E(K)$, $P \neq O$. *Then the section* (P) *is disjoint from the zero section* (O) *if and only if* x *and* y *are polynomials in* t *of degree* ≤ 2 *or* ≤ 3, *i.e., of the form*:

(3.2) $$x = gt^2 + at + b, \qquad y = ht^3 + ct^2 + dt + e,$$

with $a, b, \cdots, g, h \in k$.

By (2.1) and Theorem 2.2, we have:

Theorem 3.2. *Let* $f: S \to \boldsymbol{P}^1$ *be a rational elliptic surface.*

(i) *If* f *has no reducible fibres, there are exactly* 240 *points* $P=(x, y)$ *of the form* (3.2), *and they contain a set of* 8 *free generators of the Mordell-Weil group* $E(K)$.

(ii) *If there is only one reducible fibre $f^{-1}(v)$, we may take $v=\infty$. In case $m_\infty=2$, then there are exactly 56 points (x, y) of the form:*

(3.3) $$x=at+b, \qquad y=ct^2+dt+e,$$

and they contain a set of 7 free generators of $E(K)$.

(iii) *Similarly, if $f^{-1}(\infty)$ is the only reducible fibre and $m_\infty=3$, then there are exactly 54 points of the form (3.3), and they contain a set of 6 free generators of $E(K)$,*

(iv) *If there are only two reducible fibres, we may assume they lie over $v=0$ and ∞. In case $m_0=m_\infty=2$, then there are 12 points of the form (3.3) with $b=e=0$ and 32 points of the form (3.3) with $be\neq 0$. They contain a set of 6 free generators of $E(K)$.*

In general, for an elliptic surface $f: S\to C$, let $f^{-1}(v)^\#$ denote the smooth part of the fibre $f^{-1}(v)$, which is a commutative algebraic group. Let $sp_v: E(K)\to f^{-1}(v)^\#$ be the specialization homomorphism: for any $P\in E(K)$, $sp_v(P)$ is the unique intersection point of the section (P) with the fibre $f^{-1}(v)$.

Lemma 3.3. *Suppose that $f^{-1}(\infty)$ is a singular fibre of type II (a rational curve with a cusp). Then $f^{-1}(\infty)^\#$ is the additive group $\boldsymbol{G}_a$, and the specialization map $sp_\infty: E(K)\to \boldsymbol{G}_a(k)$ takes the point P of the form (3.2) to $sp_\infty(P)=g/h$ (note $g\cdot h\neq 0$ in this case).*

Lemma 3.4. *In case $E(K)\simeq E_8$, the 240 minimal sections are mapped to 240 distinct values of k under the specialization map sp_∞.*

4. Examples. Let us give some explicit examples, chosen mostly from elliptic surfaces of Delsarte type (cf. [10], [11]). These examples alone provide rather interesting extensions of cyclotomic fields, having the Galois representation of type $E_8, \cdots$ on the Mordell-Weil lattice (cf. the part III). Moreover, in such an extension, we can *explicitly* write down the minimal sections (and hence the generators) of the Mordell-Weil group, or equivalently, the exceptional curves of the first kind on the elliptic surface. If we change the viewpoint, we have a systematic method to realize the lattice of type E_8, E_6, etc. in certain cyclotomic fields or their extensions.

Assume for simplicity that $\mathrm{char}(k)=0$, and let $\gamma\in k$, $\gamma\neq 0$, unless otherwise stated. As before, let $K=k(t)$, and $\zeta_m=e^{2\pi i/m}$.

(4.1) $$y^2=x^3+\gamma x+t^5 \qquad \text{(cf. [11])}$$

The singular fibres are all irreducible: indeed, there are one of type II at $t=\infty$, ten of type I_1 (a rational curve with a node) at $t\neq\infty$. Hence $r=8$, and $E(K)\simeq E_8$. The specialization map sp_∞ gives an isomorphism of $E(K)$ onto $\boldsymbol{Z}[\zeta_{20}](\gamma/G)^{1/20}$, where $G\in\boldsymbol{Q}(\zeta_{20})$ is independent of γ and $(\gamma/G)^{1/20}$ is a fixed 20-th root of γ/G. The 240 values corresponding to the minimal sections are given by

$$\eta_i\zeta_{20}^j(\gamma/G)^{1/20} \qquad (1\leq i\leq 12,\ 1\leq j\leq 20),$$

where η_i are certain units ($i\leq 8$) or 5-times units ($i>8$) in $\boldsymbol{Q}(\zeta_{20})$.

(4.2) $$y^2=x^3+\gamma+t^5$$

(4.3) $$y^2=x^3+\gamma t+t^5$$

In both cases, there are six singular fibres of type II, and hence $E(K)\simeq$

E_8. The specialization map sp_∞ gives an isomorphism

$$E(K) \simeq \boldsymbol{Z}[\zeta_{30}](\gamma/G)^{1/30} \quad \text{or} \quad \boldsymbol{Z}[\zeta_{24}](\gamma/G)^{1/24}$$

where $G \in \boldsymbol{Q}(\zeta_{30})$ or $\boldsymbol{Q}(\zeta_{24})$. In all three examples, the isomorphism is compatible with the action of the Galois group. Geometrically, it shows that the monodromy around $\gamma=0$ is finite and of order 20, 30 or 24. Note that, for $\gamma=0$, the Mordell-Weil group degenerates to O.

(4.4) $$y^2 = x^3 + \gamma x + t^6$$

All the singular fibres are of type I_1 and hence $r=8$. This example is treated in [8], where 8 sections generating a subgroup of index 4 in $E(K)$ are given. Of course, one can give full generators.

(4.5) $$y^2 = x^3 + \gamma t x + t^5$$

This has a singular fibre of type III (two smooth rational curves tangent at a common point) over $t=0$, and other fibres are irreducible. Hence $r=7$ and $E(K) \simeq E_7^*$. The specialization map sp_∞: $E(K) \to \boldsymbol{Z}[\zeta_{14}](-\gamma)^{1/14}$ is surjective, with kernel of rank 1, but it maps the 56 minimal sections to distinct values.

(4.6) $$y^2 = x^3 + \gamma t^2 + t^5$$

There is a singular fibre of type IV (three smooth rational curves meeting at one point transversally) over $t=0$, one of type II over $t=\infty$ and six of type I_1. Hence $r=6$ and $E(K) \simeq E_6^*$. The specialization map sp_0 gives an isomorphism:

$$E(K) \simeq \boldsymbol{Z}[\zeta_9](\gamma/G)^{1/18} \qquad (G \in \boldsymbol{Q}(\zeta_9)).$$

(4.7) $$y^2 = x^3 + (t+t^3)x + \gamma t^3$$

There are two reducible fibres over $t=0$ and ∞, both of type III, and other fibres are irreducible. Hence $r=6$ and $E(K) \simeq D_6^*$.

(4.8) $$y^2 + \gamma x y = x^3 + t^5$$

This has a singular fibre of type I_5 (five smooth rational curves forming a pentagon) over $t=0$, one of type II over $t=\infty$. We have $r=4$ and $E(K) \simeq A_4^*$. Further the specialization map gives an isomorphism sp_0: $E(K) \simeq \boldsymbol{Z}[\zeta_5] \cdot \gamma^{1/5}$.

(4.9) $$y^2 + \gamma x y = x^3 + t^m$$

This elliptic surface is rational only for $m \leq 6$, and a $K3$ surface for $7 \leq m \leq 12$. For $m=8$, 9, 10 or 12, these elliptic $K3$ surfaces give examples for which the Mordell-Weil lattice $E(K)/(\mathrm{tor})$ is not equal to the dual lattice of the narrow Mordell-Weil lattice. Hence the assumption of Theorem 1.4 that the Néron-Severi lattice be unimodular is not superfluous.

Reference

[0] T. Shioda: Mordell-Weil lattices and Galois representation. I. Proc. Japan Acad., **65**A, 268–271 (1989).