80. *Properties of Certain Integral Operator*

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1. **Introduction.** Let $\mathcal{A}_n$ denote the class of functions of the form

$$f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathcal{N}=\{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z:|z|<1\}$.

A function $f(z)$ in the class $\mathcal{A}_n$ is said to be a member of the class $\mathcal{A}_n(\alpha)$ if it satisfies

$$\left|\frac{f(z)}{z}-1\right|<1-\alpha \qquad (z \in \mathcal{U}) \tag{1.2}$$

for some α $(0 \leqq \alpha<1)$.

Let the functions $f(z)$ and $g(z)$ be analytic in the unit disk $\mathcal{U}$. Then the function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $w(z)$ analytic in $\mathcal{U}$, with $w(0)=0$ and $|w(z)|<1$ $(z \in \mathcal{U})$, such that

$$f(z)=g(w(z)) \qquad (z \in \mathcal{U}). \tag{1.3}$$

We denote this subordination by

$$f(z) \prec g(z). \tag{1.4}$$

In particular, if $g(z)$ is univalent in $\mathcal{U}$, then the subordination (1.4) is equivalent to $f(0)=g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$ (cf. [2]).

This concept of subordination can be traced to Lindelöf [5], but Littlewood ([6], [7]) and Rogosinski ([10], [11]) introduced the term and discovered the basic properties.

For a function $f(z)$ belonging to the class $\mathcal{A}_n$, we define the generalized Libera integral operator J_c by

$$J_c(f(z))=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \qquad (c \geqq 0). \tag{1.5}$$

The operator J_c, when $c \in \mathcal{N}$, was introduced by Bernardi [1]. In particular, the operator J_1 was studied earlier by Libera [4] and Livingston [8].

2. **Properties of the operator J_c.** In order to derive our results, we have to recall here the following lemma due to Miller and Mocanu [9] (also Jack [3]).

Lemma. *Let the function*

$$w(z)=b_n z^n+b_{n+1}z^{n+1}+\cdots \qquad (n \in \mathcal{N}) \tag{2.1}$$

be regular in the unit disk $\mathcal{U}$ *with* $w(z)\not\equiv 0$ $(z \in \mathcal{U})$. *If* $z_0=r_0e^{i\theta_0}$ $(r_0<1)$ *and*

$$|w(z_0)|=\max_{|z|\leqq r_0}|w(z)|, \tag{2.2}$$

then

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(2.3) $$z_0 w'(z_0) = m w(z_0),$$
where m is real and $m \geqq n \geqq 1$.

Applying the above lemma, we prove

Theorem 1. *If a function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{A}_n(\alpha)$, *then*

(2.4) $$\frac{J_c(f(z))}{z} \prec 1 + \frac{(1-\alpha)z}{n+1}.$$

Proof. It is clear for $f(z) \equiv z$ $(z \in \mathcal{U})$. Then we assume that $f(z) \not\equiv z$ $(z \in \mathcal{U})$. Define the function $w(z)$ by

(2.5) $$\frac{J_c(f(z))}{z} = 1 + \frac{(1-\alpha)w(z)}{n+1}$$

for $f(z) \in \mathcal{A}_n(\alpha)$, then we see that $w(z) = b_n z^n + b_{n+1} z^{n+1} + \cdots$ is regular in $\mathcal{U}$ and $w(z) \not\equiv 0$ $(z \in \mathcal{U})$. Note that

(2.6) $$(J_c f(z))' = -c \frac{J_c(f(z))}{z} + (c+1) \frac{f(z)}{z}.$$

Therefore, if follows from (2.5) and (2.6) that

(2.7) $$\frac{f(z)}{z} - 1 = \frac{1-\alpha}{n+1} \left(w(z) + \frac{z w'(z)}{c+1} \right).$$

Suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, with the aid of Lemma, we have

(2.8) $$\left| \frac{f(z_0)}{z_0} - 1 \right| = \frac{1-\alpha}{n+1} \left| w(z_0) + \frac{z_0 w'(z_0)}{c+1} \right|$$
$$= \frac{1-\alpha}{n+1} \left(1 + \frac{m}{c+1} \right) \geqq \frac{(1-\alpha)(n+c+1)}{(n+1)(c+1)} \geqq 1-\alpha,$$

which contradicts that $f(z) \in \mathcal{A}_n(\alpha)$. This shows that $|w(z)| < 1$ for all $z \in \mathcal{U}$, that is, that

$$\frac{J_c(f(z))}{z} \prec 1 + \frac{(1-\alpha)z}{n+1}.$$

Taking $c=0$ in Theorem 1, we have

Corollary 1. *If* $f(z) \in \mathcal{A}_n(\alpha)$, *then*

(2.9) $$\frac{1}{z} \int_0^z \frac{f(t)}{t} dt \prec 1 + \frac{(1-\alpha)z}{n+1}.$$

Next, we have

Theorem 2. *If a function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{A}_n(\alpha)$, *then*

(2.10) $$\operatorname{Re}\left\{ e^{i\beta} \frac{J_c(f(z))}{z} \right\} > 0 \qquad (z \in \mathcal{U}),$$

where

(2.11) $$|\beta| \leqq \frac{\pi}{2} - \sin^{-1}\left(\frac{1-\alpha}{n+1} \right).$$

The bound of $|\beta|$ *is best possible for the function* $f(z)$ *defined by*

$$f(z)=z+\frac{(1-\alpha)(n+c+1)}{(n+1)(c+1)}z^{n+1}. \tag{2.12}$$

Proof. By virtue of Theorem 1, we see that

$$\left|\frac{J_c(f(z))}{z}-1\right|<\frac{1-\alpha}{n+1} \qquad (z\in \mathcal{U}). \tag{2.13}$$

Therefore, it follows from (2.13) that

$$\mathrm{Re}\left\{e^{i\beta}\frac{J_c(f(z))}{z}\right\}>0 \qquad (z\in \mathcal{U})$$

for

$$|\beta|\leqq\frac{\pi}{2}-\sin^{-1}\left(\frac{1-\alpha}{n+1}\right).$$

Further, the bound of $|\beta|$ is best possible for the function $f(z)\in\mathcal{A}_n(\alpha)$ defined by

$$\frac{J_c(f(z))}{z}=1+\frac{(1-\alpha)z^n}{n+1}, \tag{2.14}$$

which is equivalent to (2.12).

Letting $c=0$ in Theorem 2, we have

Corollary 2. *If $f(z)\in\mathcal{A}_n(\alpha)$, then*

$$\mathrm{Re}\left\{\frac{e^{i\beta}}{z}\int_0^z\frac{f(t)}{t}dt\right\}>0 \qquad (z\in \mathcal{U}), \tag{2.15}$$

where

$$|\beta|\leqq\frac{\pi}{2}-\sin^{-1}\left(\frac{1-\alpha}{n+1}\right).$$

The bound of $|\beta|$ is best possible for the function $f(z)$ defined by

$$f(z)=z+(1-\alpha)z^{n+1}. \tag{2.16}$$

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