79. *Negativity and Vanishing of Microfunction Solution Sheaves at the Boundary*

By Nobuyuki TOSE and Motoo UCHIDA
Department of Mathematics, Faculty of Sciences, University of Tokyo

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Introduction. Let M be a real analytic manifold with a complexification X. Let V be a $\mathbb{C}^\times$-conic involutive submanifold of $\mathring{T}^*X(=T^*X\setminus X)$, and let $\mathfrak{M}$ be a coherent $\mathcal{E}_X$-module with constant multiplicity along V. Moreover let Ω be an open subset of M with real analytic boundary $N=\partial\Omega$. The aim of this note is to give vanishing theorems for the cohomology groups of the complex $\boldsymbol{R}\,\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M}, \mathcal{C}_{\Omega|X})$ where $\mathcal{C}_{\Omega|X}$ is the complex of microfunctions at the boundary introduced by P. Schapira [8] (see §1.1 for the definition).

The vanishing of the complex $\boldsymbol{R}\,\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M}, \mathcal{C}_M)$ has been studied by M. Sato *et al.* [6], M. Kashiwara [3] and Kashiwara-Schapira [5], and we study in this note an analogous problem at the boundary.

1. Preliminary and a lemma. 1.1. Let M be a real analytic manifold of dimension n with a complexification X, and let Ω be an open subset of M with real analytic boundary $N=\partial\Omega$.

The cotangent bundle T^*X of X is endowed with the sheaf $\mathcal{E}_X$ of microdifferential operators of finite order. Refer to M. Sato *et al.* [6] and P. Schapira [7] for detailed account of $\mathcal{E}_X$. Let $T^*_\Omega X$ denote the micro-support of $\mathbb{Z}_\Omega$ due to [4], and let $\mathcal{C}_{\Omega|X}$ be the complex of microfunctions along $T^*_\Omega X$ introduced by P. Schapira [8]. With the bifunctor $\mu\mathrm{hom}(\cdot,\cdot)$ constructed by Kashiwara-Schapira [4], the complex $\mathcal{C}_\Omega|_X$ is explicitly given by

$$\mathcal{C}_{\Omega|X}=\mu\mathrm{hom}(\mathbb{Z}_\Omega, \mathcal{O}_X)\otimes or_M[n]$$

where or_M denotes the orientation sheaf on M.

1.2. We follow the notation in §1.1. Let V be a $\mathbb{C}^\times$-conic involutive submanifold of $\mathring{T}^*X$. We recall the Levi form $\mathcal{L}_\Lambda(V)(p)$ of V along $\Lambda=T^*_MX$ at $p\in\Lambda\cap V$. Take a system of functions $(f_1,\cdots,f_l)$ so that $V=\{q\in\mathring{T}^*X;\ f_1(q)=\cdots=f_l(q)=0\}$ locally in a neighborhood p. Then $\mathcal{L}_\Lambda(V)(p)$ denotes the Hermitian form given by the matrix $(\{f_i, f_j^c\})_{1\le i,j\le l}$. Here f_j^c is the complex conjugate of f_j and $\{\cdot,\cdot\}$ is the Poisson bracket. We remark that the signature of $\mathcal{L}_\Lambda(V)(p)$ is independent of the choice of $(f_1,\cdots,f_l)$. Refer to M. Sato *et al.* [6] and Kashiwara-Schapira [5].

1.3. Let X be a C^∞ manifold. Then $D^b(X)$ denotes the derived category of the category of bounded complexes of sheaves on X. For $F\in Ob(D^b(X))$, SS(F) is its micro-support. Let Z_1 and Z_2 be two subsets in X. Then $C(Z_1,Z_2)$ is the tangent cone for the pair (Z_1,Z_2). Refer to Kashiwara-Schapira [4] for all in this §1.3.

1.4. Now we give a lemma about the micro-support of $\mathcal{C}_{\Omega|X}$-solution complex to a system of microdifferential equations with constant multiplicity. Let M, X, Ω and N as in § 1.1, and V be as in § 1.2. Let $\mathfrak{M}$ be a coherent $\mathcal{E}_X$-module defined in a neighborhood of $p \in V$. We assume that $\mathfrak{M}$ is with constant multiplicity along V. Then we have

Lemma. $\mathrm{SS}(\boldsymbol{R}\,\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M}, \mathcal{C}_{\Omega|X})) \subset C\,(\mathrm{Char}(\mathfrak{M}), T^*_\Omega X)$.

Proof. In view of [6; Th. 5.3.7, Chap. II], we may assume that $\mathfrak{M}$ is simple characteristic along V. Then on account of [6; Th. 5.1.2, Chap. II], we can find a quantized contact transformation (φ, Φ), through which we have an isomorphism

$$\varphi_*\mathfrak{M} \simeq \mathcal{E}_X/(\mathcal{E}_X D_1 + \cdots + \mathcal{E}_X D_l) \quad (= \mathcal{E}_X \otimes_{\pi_X^{-1}\mathcal{D}_X} \mathfrak{M}_0).$$

Here π_X is the natural projection $\pi_X : T^*X \to X$, and $\mathfrak{M}_0$ is a coherent $\mathcal{D}_X$-module $\mathcal{D}_X/(\mathcal{D}_X D_1 + \cdots + \mathcal{D}_X D_l)$. Moreover by the theory of [4; Chap. 11] (cf. also [11]), there exist $F_{\Lambda_0} \in Ob(D^b(X))$ $(\Lambda_0 = \varphi(T^*_\Omega X))$ and an isomorphism

$$\varphi_*\mathcal{C}_{\Omega|X} \simeq \mu\mathrm{hom}(F_{\Lambda_0}, \mathcal{O}_X)$$

in a neighborhood of $\varphi(p)$. Hence we have

$$\begin{aligned}\mathrm{SS}(\boldsymbol{R}\,\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M}, \mathcal{C}_{\Omega|X})) &= \varphi^{-1}\mathrm{SS}(\boldsymbol{R}\,\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathfrak{M}_0, \mu\mathrm{hom}(F_{\Lambda_0}, \mathcal{O}_X)))\\ &\subset \varphi^{-1}C(\mathrm{Char}(\mathfrak{M}_0), \Lambda_0) = C(\mathrm{Char}(\mathfrak{M}), T^*_\Omega X).\end{aligned}$$

Remark that $\mathrm{SS}(\cdot)$ for the solution complex to the $\mathcal{D}_X$-module $\mathfrak{M}_0$ can be easily estimated as above (see [4]). Q.E.D.

2. Main theorems. Let M, X, Ω and N be as in § 1.1. Then we give

Theorem 1. *Let $p \in \mathring{T}^*_M X$ with $\pi_X(p) \in N$, and let $V = \{q \in \mathring{T}^*X;\ f(q) = 0\}$ be given by a homogeneous holomorphic function f satisfying the condition*

(1) $$\{f, f^c\}(p) < 0.$$

Assume that there exists a homogeneous holomorphic function ψ for which the following conditions (2), (3), (4), (5) *are satisfied.*

(2) $d\psi \wedge \omega_X \neq 0$ *at* p. (ω_X *is the canonical* 1-*form of* T^*X.)

(3) $$V \cap \bar{V} \subset \{\psi = 0\}.$$

(4) $$\mathrm{Im}\,\psi|_{T^*_M X} = 0.$$

(5) $\pi_X^{-1}(\Omega) \cap T^*_M X \subset \{\psi > 0\}$ *in a neighborhood of* p.

Let $\mathfrak{M}$ be coherent $\mathcal{E}_X$-module with constant multiplicity along V defined in a neighborhood of p. Then we have

$$\mathrm{H}^0\,\boldsymbol{R}\,\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M}, \mathcal{C}_{\Omega|X})_p = 0.$$

Proof. First we give a geometric argument in $S^*_M X$ and in $(S^*_M X)^C$. We take symplectic coordinates of $S^*_M X$ as $(x_1, \cdots, x_n; p_1, \cdots, p_{n-1})$. We set in $(S^*_M X)^C$

$$Y = \{\psi = 0\}.$$

Then we may assume

$$f + f^c = 0 \qquad \text{on } Y.$$

By [6; Lemma 2.3.3, Chap. III], we can find a holomorphic function $\phi\ (\neq 0)$ real valued on $S^*_M X$ such that $\{\phi f, \phi^c f^c\} = -1$. Thus we may assume from the beginning that

$$\{f, f^c\} = -1 \text{ on } S^*_M X, \text{ and } f + f^c = 0 \text{ on } Y.$$

Moreover making a change of symplectic coordinates of S_M^*X, we may write f and Y as

$$f=p_1+\sqrt{-1}\,x_1 \qquad Y=\{z_1=0\}$$

with the complexified coordinate z_1 of x_1. (See [6; Lemma 2.3.9, Chap. III].) Thus by finding a real quantized contact transformation, we may assume

$$f=\zeta_1+\sqrt{-1}\,z_1\zeta_n, \qquad \psi=z_1, \qquad p=(0\,;\sqrt{-1}\,dx_n).$$

Here we take a system of coordinates of T^*X as $(z\,;\zeta\cdot dz)$ and that of T_M^*X as $(x\,;\sqrt{-1}\xi\cdot dx)$. In this situation, a direct calculation gives

$$(6) \qquad -d\psi \notin C(\{f=0\},\ \{\psi\geq 0\}\cap T_M^*X).$$

Thus the fact (6) holds also in the general case.

Now we take a real analytic function φ such that $\Omega=\{\varphi>0\}$ in a neighborhood of $\pi_X(p)$, and $d\varphi\neq 0$ on N. Since $\pi_X^{-1}(\Omega)\cap T_M^*X\subset\{\psi>0\}\cap T_M^*X$ in a neighborhood of p and

$$-d\varphi(p)=-d\psi(p) \qquad \text{mod}\,(T^*_{T_M^*X}T^*X)_p\,(\simeq T_pT_M^*X),$$

we get

$$(7) \qquad -d\varphi \notin C(\{f=0\},\ \overline{\pi_X^{-1}(\Omega)}\cap T_M^*X).$$

This implies (cf. [10])

$$(8) \qquad -d\varphi \notin C(\{f=0\},\ T_\Omega^*X).$$

Thus on account of the lemma in § 1.4, we deduce

$$\boldsymbol{R}\Gamma_{\{\varphi\leq 0\}}\boldsymbol{R}\,\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M},\mathcal{C}_{\Omega|X})_p=0.$$

Hence we obtain the isomorphism

$$\boldsymbol{R}\,\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M},\mathcal{C}_{\Omega|X})_p\xleftarrow{\ \sim\ }\boldsymbol{R}j_*j^{-1}\boldsymbol{R}\,\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M},\mathcal{C}_M)_p \quad (j:\pi_X^{-1}(\Omega)\cap T_M^*X\hookrightarrow T_M^*X).$$

On the other hand, by [6; Th. 2.3.10, Chap. III] (cf. [5]), we have $\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M},\mathcal{C}_M)=0$. Hence we conclude

$$\mathrm{H}^0\boldsymbol{R}\,\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M},\mathcal{C}_{\Omega|X})_p\xleftarrow{\ \sim\ }j_*j^{-1}\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M},\mathcal{C}_M)_p=0. \qquad \text{Q.E.D.}$$

Theorem 2. *Let p, V, Ω be as in Theorem 1. Let W $(\subset V)$ be a $\mathbb{C}^\times$-conic involutive variety in $\mathring{T}^*X$ through p with q (≥ 1) negative eigenvalues of $\mathcal{L}_\Lambda(W)(p)$. Let $\mathfrak{M}$ be a coherent $\mathcal{E}_X$-module with constant multiplicity along W defined in a neighborhood of p. Then we have*

$$\mathrm{H}^j\boldsymbol{R}\,\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M},\mathcal{C}_{\Omega|X})_p=0 \qquad (j<q).$$

Moreover if $\mathcal{L}_\Lambda(W)(p)$ is non-degenerate, then we have the vanishing of the left-hand side for $j\neq q$.

Proof. This theorem can be proved in the same way as Theorem 1 if we remark

$$\underline{\mathrm{Ext}}^j_{\mathcal{E}_X}(\mathfrak{M},\mathcal{C}_M)=0 \qquad (j<q) \text{ in a neighborhood of } p.$$

(See [6; Th. 2.3.10, Chap. III].) Moreover if $\mathcal{L}_\Lambda(W)(p)$ is non-degenerate, M. Sato *et al.* [6; Th. 2.3.6, Chap. III] have shown that in a neighborhood of p, $\underline{\mathrm{Ext}}^j_{\mathcal{E}_X}(\mathfrak{M},\mathcal{C}_M)=0$ $(j\neq q)$ and $\underline{\mathrm{Ext}}^q_{\mathcal{E}_X}(\mathfrak{M},\mathcal{C}_M)$ is conically flabby. Hence the proof of Theorem 1 gives

$$\mathrm{H}^i\boldsymbol{R}\,\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M},\mathcal{C}_{\Omega|X})_p\simeq\boldsymbol{R}^{i-q}j_*j^{-1}\,\underline{\mathrm{Ext}}^q_{\mathcal{E}_X}(\mathfrak{M},\mathcal{C}_M)_p=0 \quad (i\neq q). \qquad \text{Q.E.D.}$$

Next we give a generalization of Theorem 1.

Theorem 3. *Let $p\in\mathring{T}_M^*X$ with $\pi_X(p)\in N$, and let W be a $\mathbb{C}^\times$-conic involutive variety of codimension d with $p\in W$ and $\mathcal{L}_\Lambda(W)(p)<0$. Assume that there exists a homogeneous holomorphic function ψ with the proper-*

ties;

(9) $$d\psi\wedge\omega_X\neq 0 \quad at\ p,$$

(10) $$\operatorname{Im}\psi|_{T^*_MX}=0,$$

(11) $$W\cap\overline{W}\subset\{\psi=0\},$$

(12) $$\pi_X^{-1}(\Omega)\cap T^*_MX\subset\{\psi>0\}\ in\ a\ neighborhood\ of\ p.$$

Let $\mathfrak{M}$ be a coherent $\mathcal{E}_X$-module with constant multiplicity along W. Then we have

$$\mathrm{H}^j\boldsymbol{R}\,\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathfrak{M},\mathcal{C}_{\Omega|X})_p=0 \quad (j\neq d).$$

Proof. We write W as $W=\{f_1=\cdots=f_d=0\}$. Since $\mathcal{L}_\Lambda(W)(p)$ is non-degenerate, we have $df_1\wedge\cdots\wedge df_d\wedge df_1^c\wedge\cdots\wedge df_d^c\wedge\omega_X\neq 0$. (This shows in particular that W is non-singular.) Next we remark that ψ is real simple. Then taking into account of the assumption (11), we can find homogeneous holomorphic functions $\{a_j\}_{1\leq j\leq d}$ with $a_j\neq 0$ for some j which satisfy

$$\psi=\sum_{j=1}^d(a_jf_j+a_j^cf_j^c).$$

In this situation, put $V=\left\{f:=\sum_{j=1}^d a_jf_j=0\right\}$. Then we have

$$\{f,f^c\}=\sum_{1\leq j,k\leq d}a_ja_k^c\{f_j,f_k^c\}<0,$$

which makes it possible to apply Theorem 2. Q.E.D.

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