76. First Order Rational Differential Equations Depending Transcendentally on Arbitrary Constants

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0. In [3] Painlevé gives an example of first order rational differential equation whose general solution depends transcendentally on arbitrary constants. This example, however, as will be seen in later, is defined essentially over the complex number field. The aim of this note is to get an example defined over the field containing nonconstant functions without separable variables. To this end it will be necessary to seize some of notions introduced by Painlevé from the viewpoint of differential algebra.

Let K be a differential field of characteristic 0 with a single differentiation $'$. In what follows every differential field extension of K will be regarded as differential subfields of a fixed universal differential field extension of K. Let R be a differential field extension of K and a finitely generated field extension of K. We say that R depends algebraically on arbitrary constants if there exists a differential field extension E of K such that E and R are free over K and $m(R:E)=[ER:EC_{ER}]$ is finite, where C_L for a differential field L denotes the field of constants of L. If this is case, by $m(R)$ we denote the minimum of such numbers $m(R;E)$. Then there exists an intermediate differential field S between R and K such that $m(R)=[R:S]$ and $m(S)=1$ provided K is algebraically closed (see [2]). We remark that if we consider a new differentiation $*$ in R by $u^*=au'$ for any u in R with a fixed nonzero a in K the property of algebraic dependence on arbitrary constants will be left unaltered, because in the above definition to be constant with respect to $'$ is the same as be so with respect to $*$. The number $m(R)$ corresponds to the number of branches of general solution around a movable singularity which was investigated by Painlevé.

1. **Lemma.** *Let R be a differential algebraic function field of one variable over K. If there exists a finite chain of differential field extensions of K: $K=F_0\subseteq F_1\subseteq\cdots\subseteq F_m$ such that $R\subseteq F_m$ and for each i F_i is algebraic extension of F_{i-1} or a differential algebraic function field over F_{i-1} depending algebraically on arbitrary constants then R depends algebraically on arbitrary constants.*

Proof. Let m be the minimum index for which R is contained is some finite algebraic extension F of F_m. Then F is a differential algebraic function field over F_{m-1} depending algebraically on arbitrary constants. Hence there exists a differential field extension E of K such that E and F are free over K and $m(F:E)$ is finite. Since C_{EF} and ER are linearly disjoint over

C_{ER}, it follows $\mathrm{tr.deg}\, C_{EF}/C_{ER} = \mathrm{tr.deg}\, EF/ER = \mathrm{tr.deg}\, EF/E - 1$ therefore $[ER : EC_{ER}]$ is finite. This shows R depends algebraically on arbitrary constants.

Theorem 1. *Let y be the general solution of the rational differential equation $y' = f(y)$, where f is a nonzero rational function in $\boldsymbol{C}(y)$. Let $\boldsymbol{C} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_m$ be the same as in Lemma 1. Suppose that y is contained in F_m. Then there is a nonconstant rational function u satisfying fu_y or $fu_y/u \in \boldsymbol{C}$.*

Proof. The field $R = \boldsymbol{C}(y)$ is regarded as a differential field extension of $\boldsymbol{C}$. Lemma 1 shows R depends algebraically on arbitrary constants. Therefore there is an intermediate differential field S between R and $\boldsymbol{C}$ for which $[R : S]$ is finite and $m(S) = 1$. S has no movable singularity (cf. [1, p. 98]). By Lüroth's theorem S can be described as $S = \boldsymbol{C}(z)$. The element z satisfies a Riccati equation defined over $\boldsymbol{C}$ (cf. [1, p. 13]). Through a certain linear transformation of z we have thus the description $S = \boldsymbol{C}(u)$, u' or $u'/u \in \boldsymbol{C}$. We see $u' \neq 0$, for otherwise $f(y) = y' = 0$, which shows $y \in \boldsymbol{C}$, a contradiction. This completes the proof.

Remark. This theorem is a generalization of Proposition 2 in [4].

Now let us explain Painlevé's example of differential algebraic function field over $L = \boldsymbol{C}(x)$ which does not depend algebraically on arbitrary constants. Consider the general solution y of the following equation

$$y' = y/x(y+1), \qquad ' = d/dx.$$

Then defining a new differentiation $*$ of $L(y, y')$ by $u^* = xu'$ for $u \in L(y, y')$, according to Theorem 1 we find $\boldsymbol{C}(y, y^*)$ hence $L(y, y')$ does not depend algebraically on arbitrary constants.

2. Theorem 2. *Let K be algebraically closed. Let y be the general solution of $y' = f(y) = A(y)/B(y)$ over K, where A and B are coprime polynomials in $K[y]$ and $\deg A > \deg B + 1$. Suppose that $R = K(y)$ depends algebraically on arbitrary constants and $A(k) \neq k'B(k)$ for any $k \in K$. Let $m = m(R)$. Then $\deg A = 2m$ and $m - 1 \leqq \deg B \leqq 2(m-1)$.*

Proof. There is a intermediate differential field S between R and K which satisfies $[R : S] = m$ and $m(S) = 1$. It has no movable singularity. Let v be a normalized valuation of R. We take as a prime element $t = y - k$, $k \in K$, or $t = 1/y$. Then if $v(y) \geqq 0$, $v(t') = -v(B)$, otherwise, $v(t') = 2 - \deg A + \deg B$, where B is regarded as an element of $K[y]$. For in the first case we have $v(f) = -v(B)$ if $v(B) > 0$ and $v(t') = 0$ if $v(B) = 0$ since $f(k) - k' \neq 0$ by our assumption. In the case where $t = 1/y$ we have

$$t' = -y'/y^2 = -t^2 A(1/t)/B(1/t).$$

In any case it results $v(t') \leqq 0$. Let e be the ramification index of v with respect to S and w the restriction of v to S. A prime element u of w has the representation $u = a_0 t^e + a_1 t^{e+1} + \cdots$. Hence $v(u') = e - 1 + v(t')$. On the other hand, since S has no movable singularity, $ew(u') = v(u') \geqq 0$. Therefore $v(u') = 0$ and $v(t') = 1 - e$. Applying Hurwitz' formula for R and S, both of which have the genus 0, we have $2(m-1) = \sum_P (e_P - 1)$, where P

runs through all K-places of R and e_P denotes the ramification index of P with respect to S. From this it results $2(m-1)=\sum_v(-v(t'))$, with v normalized valuations. Therefore

$$\begin{aligned}2(m-1)&=\sum_{v(y)\geqq 0}(-v(t'))+\sum_{v(y)<0}(-v(t'))\\&=\deg B+\deg A-\deg B-2=\deg A-2.\end{aligned}$$

Therefore $\deg A=2m$. In view of this equality and the assumption $\deg A\geqq \deg B+2$, we see $\deg B\leqq 2(m-1)$. Furthermore above two results on $v(t')$ in case $v(y)<0$ imply $2+\deg B-\deg A=1-e$ or $e=2m-\deg B-1$. Since ramification indices do not exceed the degree $m=[R:S]$, it follows $\deg B\geqq m-1$. This completes the proof.

3. Theorem 3. *Let $a\neq 0$ and b be complex numbers and m be a natural number larger than 2. Let y be the general solution of $y'=y^m+ax+b$ over $\boldsymbol{C}(x)$. Then the differential function field $\boldsymbol{C}(x,y)$ over $\boldsymbol{C}(x)$ does not depend algebraically on arbitrary constants.*

Proof. According to Theorem 2 it remains only to prove that $k'\neq k^m+ax+b$ for any $k\in K$, K being the algebraic closure of $\boldsymbol{C}(x)$. Assume the converse, namely, we have a, k in K with $k'=k^m+ax+b$. Let $h=k'\in K$. Clearly $h\neq 0$. In the equation $k''=mk^{m-1}k'+a$, changing the independent variable, we have $hh^*=mk^{m-1}h+a$, where $*=d/dk$. We first show h to be integral over $\boldsymbol{C}[k]$. For let v be a normalized valuation of the function field $\boldsymbol{C}(h,k)$ with $v(k)\geqq 0$. If $v(h)<0$ then $v(h^*)<v(h)$, thereby $v(hh^*)<v(h^*)<v(h)$. On the other hand $v(hh^*)=v(mk^{m-1}h+a)\geqq\min\{(m-1)v(k)+v(h),0\}\geqq v(h)$, which is a contradiction. Hence h satisfies the equation

$$h^n+a_1h^{n-1}+\cdots+a_n=0,\qquad a_i\in\boldsymbol{C}[k].$$

Here we take n the minimum. The differentiation leads to

$$\begin{aligned}&a_1^*h^n+a_2^*h^{n-1}+\cdots+a_n^*h\\&\quad+(mk^{m-1}h+a)\{nh^{n-1}+(n-1)a_1h^{n-2}+\cdots+a_{n-1}\}=0.\end{aligned}$$

The minimality of n yields

$$a_r^*+m(n-r+1)a_{r-1}k^{m-1}+(n-r)aa_{r-2}=(a_1^*+mnk^{m-1})a_{r-1},$$

or

$$a_r^*=\{a_1^*+m(r-1)k^{m-1}\}a_{r-1}-(n-r+2)aa_{r-2},$$

where $2\leqq r\leqq n+1$ and we set $a_0=1$, $a_{n+1}=0$. Suppose that $a_1^*+m(r-1)k^{m-1}\neq 0$ for any r $(2\leqq r\leqq n+1)$. Then, since $a_2^*=(a_1^*+mk^{m-1})a_1-na$, it follows i) if $a_1=0$ then $\deg a_2=1$; ii) if $a_1\in\boldsymbol{C}$, $\neq 0$ then $\deg a_2=m$; iii) if $a_1^*\neq 0$ then $\deg a_2>\deg a_1$. Using the above recurrence relation, by induction we have $\deg a_r>\deg a_{r-1}$ $(2\leqq r\leqq n+1)$, which contradicts $a_{n+1}=0$. Therefore there exists such a number s that $2\leqq s\leqq n+2$ and $a_1^*+m(s-1)k^{m-1}=0$. Then

$$a_r^*=m(r-s)k^{m-1}a_{r-1}-(n-r+2)aa_{r-2}\qquad(2\leqq r\leqq n+2).$$

By induction it is proved that for $1\leqq r\leqq s-1$, $\deg a_r=mr$ and the leading coefficient of a_r equals $c_r=(-1)^r(s-r)\cdots(s-1)/r!$ In particular $c_{s-1}=(-1)^{s-1}$ and $c_{s-2}=(-1)^{s-2}(s-1)$. From this we have $\deg a_s=m(s-2)+1$ and a_s has the leading coefficient $(-1)^{s-1}(n-s+2)(s-1)a/\{m(s-2)+1\}$. By induction we have for $s\leqq r\leqq n+1$, $\deg a_r=m(r-2)+1$ and the leading coefficients c_r of a_r satisfy

$$\{m(s-1)+1\}\{m(s-2)+1\}c_{s+1}=(-1)^{s-1}a\{(m-1)n+s-1\},$$
$$\{m(r-)+1\}c_{r+1}=m(r-s+1)c_r \qquad (s+1\leqq r\leqq n),$$

especially $a_{n+1}\neq 0$, which is a contradiction.

References

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