69. *Applications of B(P, α)-refinability for Generalized Collectionwise Normal Spaces*

By R. H. PRICE*) and J. C. SMITH**)

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Introduction. In [5] the authors introduced the notion of $B(P,\alpha)$-refinability and used it to obtain new covering characterizations for normal and collectionwise normal spaces. In this paper we generalize various known results by obtaining analogous characterizations for collectionwise subnormal and strong-collectionwise subnormal spaces.

The properties P considered in this paper will be discrete (D), locally finite (LF), closure-preserving (CP), hereditarily closure-preserving (HCP) and point finite (PF). The symbol α will denote any countable ordinal.

Definition 1. A space X is *$B(P,\alpha)$-refinable* provided every open cover $\mathcal{U}$ of X has a refinement $\mathcal{E}=\cup\{\mathcal{E}_\beta:\beta<\alpha\}$ which satisfies i) $\{\cup\mathcal{E}_\beta:\beta<\alpha\}$ partitions X, ii) for every $\beta<\alpha$, $\mathcal{E}_\beta$ is a relatively P collection of closed subsets of the subspace $X-\cup\{\cup\mathcal{E}_\mu:\mu<\beta\}$, and iii) for every $\beta<\alpha$, $\cup\{\mathcal{E}_\mu:\mu<\beta\}$ is a closed set.

The collection $\mathcal{E}$ is often called a *$B(P,\alpha)$-refinement* of $\mathcal{U}$.

Definition 2. Let $\mathcal{G}$ be a collection of open subsets of a space X. We refer to $\mathcal{G}$ as a *θ-collection* (*almost θ-collection*) provided we can write $\mathcal{G}=\cup\{\mathcal{G}_n:n\in N\}$ such that for every $x\in X$, there exists $n(x)\in N$ such that $\mathcal{G}_{n(x)}$ is LF (PF) at x.

Definition 3. (1) Let $\mathcal{F}$ be a collection of subsets of a space X. We call $\mathcal{G}=\cup\{\mathcal{G}_n:n\in N\}$ an (*almost θ-expansion*) of $\mathcal{F}$ provided (i) $\mathcal{G}_n$ is an open expansion of $\mathcal{F}$ for every $n\in N$, and (ii) $\mathcal{G}$ is an (almost) θ-collection.

(2) A space X is (*almost*) *θ-expandable* provided every LF collection of closed subsets of X has an (almost) θ-expansion.

(3) A space X is (*almost*) *discretely-θ-expandable* provided every discrete collection of closed subsets of X has an (almost) θ-expansion.

Expandable and θ-expandable spaces have been studied in [3, 4, 9, 10].

Definition 4. A space X is collectionwise subnormal (CWSN) provided every discrete collection $\mathcal{D}$ of closed subsets of X has a pairwise disjoint G_δ-expansion which is also an almost θ-expansion of $\mathcal{D}$.

In 1979 Chaber [1] obtained the following result.

Theorem 1. *A space X is subparacompact iff X is CWSN and θ-refinable.*

Here we generalize Chaber's result by using the notion of $B(D,\omega)$-refinability.

*) Department of Mathematics, Judson College, Marion, Alabama 36756, USA.
**) Department of Mathematics, Virginia Tech, Blacksburg, Virginia 24061 USA.

Theorem 2. *Let X be a CWSN space. Let $\mathcal{U}=\{U_\alpha : \alpha \in A\}$ be an open cover of X, and $\mathcal{D}=\{D_\alpha : \alpha \in A\}$ a discrete-closed partial refinement of $\mathcal{U}$ such that $D_\alpha \subset U_\alpha$ for each $\alpha \in A$. Then there exists a G_δ-set K and a σ-discrete-closed partial refinement $\mathcal{E}$ of $\mathcal{U}$ such that $\cup\mathcal{D} \subset K \subset \cup\mathcal{E}$.*

Proof. Assume that X is CWSN. Then there exists a pairwise disjoint G_δ-expansion $Q=\{Q_\alpha=\cap\{Q(\alpha,n): n \in N\} : \alpha \in A\}$ of $\mathcal{D}$ such that $\cup\{\{Q(\alpha,n): \alpha \in A\} : n \in N\}$ is an almost θ-expansion of $\mathcal{D}$, and $D_\alpha \subset Q(\alpha,n) \subset U_\alpha$ for every $\alpha \in A$, $n \in N$.

For each $n \in N$, let $Q_n=\{Q(\alpha,n): \alpha \in A\}$, and define $C_n=X-\cup Q_n$. Then C_n and $\cup\mathcal{D}$ are disjoint closed sets, so there exist disjoint G_δ-sets H_n and K_n such that $C_n \subset H_n$, and $\cup\mathcal{D} \subset K_n$. For each n, we denote $H_n=\cap\{H(n,j): j \in N\}$ such that $H(n,j)$ is open for every $j \in N$. For every $n, j \in N$, define $\mathcal{V}(n,j)=\{H(n,j)\cap U_\alpha : \alpha \in A\}\cup Q_n$. By construction, $\mathcal{V}(n,j)$ is an open refinement of $\mathcal{U}$. Now define $E(n,j)=\{x : ord(x,\mathcal{V}(n,j))=1\}$, $\mathcal{E}(n,j)=\{E(n,j)\cap V : V \in \mathcal{V}(n,j)\}$, and $\mathcal{E}=\cup\{\mathcal{E}(n,j) : n, j \in N\}$. It should be clear that $\mathcal{E}$ is a σ-discrete-closed partial refinement of $\mathcal{U}$ since each $\mathcal{V}(n,j)$ is an open cover of X which refines $\mathcal{U}$. Define $K=\cap\{K_n : n \in N\}$, so that K is a G_δ-set and $\cup\mathcal{D}\subset K$. It remains only to show that $K \subset \cup\mathcal{E}$; so let $x \in K$. Since Q is pairwise disjoint and $\cup\{Q_n : n \in N\}$ is an almost θ-collection, there exists $n(x) \in N$ such that $ord(x, Q_{n(x)})\leq 1$. Now $K_{n(x)} \subset \cup Q_{n(x)}$ from above, so that $K \subset \cup Q_{n(x)}$. Thus, $x \in \cup Q_{n(x)}$ and hence $ord(x, Q_{n(x)})=1$. Since $H_{n(x)} \cap K_{n(x)}=\emptyset$ and $x \in K_{n(x)}$ there exists $j(x) \in N$ such that $x \notin H(n, j(x))$. Therefore, $ord(x, \mathcal{V}(n(x), j(x)))=ord(x, Q_{n(x)})=1$, implying that $x \in \cup\mathcal{E}(n(x), j(x))$, and hence $x \in \cup\mathcal{E}$.

It follows that $\cup\mathcal{D}\subset K \subset \cup\mathcal{E}$ and the proof is complete.

Remark. J. Zhu [12] has also independently obtained Theorem 3.

Theorem 3. *If X is a CWSN space, and $\mathcal{U}$ is an open cover of X which has a $B(D,\omega)$-refinement, then $\mathcal{U}$ has a σ-discrete-closed refinement.*

Proof. Let $\mathcal{E}=\cup\{\mathcal{E}_n : n \in N\}$ be a $B(D,\omega)$-refinement of $\mathcal{U}$. By induction we construct for each $n \in N$, (a_1) a G_δ-set $Q_n=\cap\{Q(n,j): j \in N\}$, and ($a_2$) a σ-discrete-closed partial refinement $\mathcal{F}_n$ of $\mathcal{U}$, such that (a_3) $\cup\{\cup\mathcal{E}_k : 1\leq k\leq n\}\subset Q_n \subset \cup\mathcal{F}_n$. It is easy to see that $\mathcal{F}=\cup\{\mathcal{F}_n : n \in N\}$ will be the desired σ-discrete closed refinement of $\mathcal{U}$, and our proof will be complete.

The above conditions are vacuously satisfied for $n=1$. Now let $n>1$ be fixed, and assume that Q_i and $\mathcal{F}_i$ have been constructed satisfying the above conditions for $1\leq i<n$. For each $j \in N$, define $\mathcal{D}(n,j)=\{E-Q(n-1,j) : E \in \mathcal{E}_n\}$. It should be clear that $\mathcal{D}(n,j)$ is a discrete-closed partial refinement of $\mathcal{U}$. By Theorem 2 above, there exists a G_δ-set $K(n,j)$ and a σ-discrete-closed partial refinement $\mathcal{F}(n,j)$ of $\mathcal{U}$ such that $\cup\mathcal{D}(n,j)\subset K(n,j)\subset \cup\mathcal{F}(n,j)$. Define $Q(n,j)=K(n,j)\cup Q(n-1,j)$, $Q_n=\cap\{Q(n,j): j \in N\}$, and $\mathcal{F}_n=\mathcal{F}_{n-1}\cup(\cup\{\mathcal{F}(n,j) : j \in N\})$.

By construction Q_n and $\mathcal{F}_n$ satisfy conditions (a_1) and (a_2) above, so it remains only to show (a_3). (i) If $x \in \cup\{\cup\mathcal{E}_k : 1\leq k<n\}$, then $x \in Q_{n-1}$. Therefore $x \in Q(n,j)$ for every j, implying $x \in Q_n$. Next, suppose that $x \in \cup\mathcal{E}_n$

and let $j' \in N$ be fixed. If $x \in Q(n-1, j')$, then $x \in Q(n, j')$. If $x \notin Q(n-1, j')$, then $x \in E - Q(n-1, j')$ for some $E \in \mathcal{E}_n$ and hence $x \in \cup \mathcal{D}(n, j')$. Therefore, $x \in K(n, j') \subset Q(n, j')$. Thus, $x \in Q_n$, and it follows that $\cup\{\cup \mathcal{E}_k : 1 \leq k \leq n\} \subset Q_n$.

(ii) Now let $x \in Q_n$. If $x \in Q(n-1, j)$ for all $j \in N$, then $x \in Q_{n-1}$ so that $x \in \cup \mathcal{F}_{n-1}$. Suppose there exists some $j' \in N$ such that $x \notin Q(n-1, j')$. Then $x \in K(n, j')$ and so $x \in \cup \mathcal{F}(n, j')$. Since $\cup \mathcal{F}_{n-1} \subset \cup \mathcal{F}_n$ and $\cup \mathcal{F}(n, j) \subset \cup \mathcal{F}_n$ for each j, it follows that $x \in \cup \mathcal{F}_n$. Therefore $Q_n \subset \cup \mathcal{F}_n$, so ($a_3$) holds.

Theorem 4. *Let X be a space with the property that every open cover of X which has a $B(D, \omega)$-refinement also has a σ-cushioned refinement. Then X is CWSN.*

Proof. Let $\mathcal{D} = \{D_\alpha : \alpha \in A\}$ be a discrete collection of closed subsets of X. Define $\mathcal{U}_1 = \{U_\alpha = X - \cup\{D_\beta : \beta \neq \alpha\} : \alpha \in A\}$, $\mathcal{U}_2 = \{X - \cup \mathcal{D}\}$, and $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$.

Now every $x \in \cup \mathcal{D}$ has order 1 with respect to $\mathcal{U}_1$, and every $x \notin \cup \mathcal{D}$ has order 1 with respect to $\mathcal{U}_2$. By Theorem 2 of [5], $\mathcal{U}$ has a $B(D, \omega)$-refinement. Let $\mathcal{F} = \cup\{\mathcal{F}_n = (\{H_n\} \cup \{F(n, \alpha) : \alpha \in A\}) : n \in N\}$ be a σ-cushioned refinement of $\mathcal{U}$ such that $H_n \subset X - \cup \mathcal{D}$, and $F(n, \alpha) \subset U_\alpha$ for every $n \in N$, $\alpha \in A$. For every $n \in N$, $\alpha \in A$, define $W(n, \alpha) = X - cl(\cup \mathcal{F}_n)$ if $F(n, \alpha) \cap D_\alpha = \emptyset$, and $W(n, \alpha) = X - cl(\cup(\mathcal{F}_n - \{F(n, \alpha)\}))$ if $F(n, \alpha) \cap D_\alpha \neq \emptyset$.

For each $n \in N$ recall that $\mathcal{F}_n$ is cushioned in $\mathcal{U}$, and that $U_\beta \cap D_\alpha = \emptyset$ whenever $\beta \neq \alpha$. It follows that $\mathcal{W}_n = \{W(n, \alpha) : \alpha \in A\}$ is an open expansion of $\mathcal{D}$ such that $D(n, \alpha) \subset W(n, \alpha)$ for each $\alpha \in A$. Hence, $\mathcal{W} = \{W_\alpha = \cap\{W(n, \alpha) : n \in N\} : \alpha \in A\}$ is a G_δ-expansion of $\mathcal{D}$.

To see that $\mathcal{W}$ is pairwise disjoint and that $\cup\{\mathcal{W}_n : n \in N\}$ is an almost θ-expansion of $\mathcal{D}$, it suffices to show for each $x \in X$ that there exists some $n(x) \in N$ such that $ord(x, \mathcal{W}_{n(x)}) \leq 1$. Now let $x \in X$ be fixed. Since $\mathcal{F}$ covers X, there exists some $n(x) \in N$ such that either (i) $x \in F(n(x), \gamma)$ for some $\gamma \in A$, or (ii) $x \in H_{n(x)}$. In either case it is easy to see that $ord(x, \mathcal{W}_{n(x)}) \leq 1$, and the proof is complete.

We now have the following from Theorem 3 and Theorem 4 above.

Corollary. *For any space X, the following are equivalent.*

(a) *X is CWSN.*

(b) *Every open cover of X which has a $B(D, \omega)$-refinement also has a σ-discrete-closed refinement.*

(c) *Every open cover of X which has a $B(D, \omega)$-refinement also has a ρ-LF-closed refinement.*

(d) *Every open cover of X which has a $B(D, \omega)$-refinement also has a σ-HCP-closed refinement.*

(e) *Every open cover of X which has a $B(D, \omega)$-refinement also has a σ-CP-closed refinement.*

(f) *Every open cover of X which has a $B(D, \omega)$-refinement also has a σ-cushioned refinement.*

Strong-Collectionwise subnormal spaces.

Definition 5. A space X is *strong-collectionwise subnormal* provided every discrete collection $\mathcal{D}$ of closed subsets of X has a pairwise disjoint G_δ-expansion which is also a θ-expansion of $\mathcal{D}$.

Examples. (1) Let S be the Sorgenfrey line and $X=S\times S$. It is well-known that X is a subparacompact, Tychonoff space which is not metacompact. Thus X is not strong-CWSN from Theorem 6 below.

(2) Let X be any countably infinite set with the cofinite topology. Then X is a strong-CWSN T_1 space which is not normal.

We now show that in the presence of strong-CWSN, subparacompact spaces are indeed metacompact.

Theorem 5. *Let X be a discretely-θ-expandable space. If $\mathcal{U}$ is any collection of open subsets of X, and $\mathcal{D}=\{D_\alpha: \alpha\in A\}$ is a discrete-closed partial refinement of $\mathcal{U}$, then $\mathcal{U}$ has a σ-PF-open partial refinement $\mathcal{V}$ such that $\cup\mathcal{D}\subset\cup\mathcal{V}$.*

Proof. Since X is discretely-θ-expandable, there exists a θ-expansion $\mathcal{G}=\cup\{\mathcal{G}_n=\{G(\alpha,n): \alpha\in A\}: n\in N\}$ of $\mathcal{D}$ such that $D_\alpha\subset G(\alpha,n)$ for every $\alpha\in A$, $n\in N$, and $\mathcal{G}$ is a partial refinement of $\mathcal{U}$. For each $n\in N$, define $V_n=\{x: \mathcal{G}_n \text{ is a } LF \text{ at } x\}$, $\mathcal{V}_n=\{\mathcal{V}_n\cap G(\alpha,n): \alpha\in A\}$, and $\mathcal{V}=\cup\{\mathcal{V}_n: n\in N\}$.

By construction, $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ such that $\cup\mathcal{D}\subset\cup\mathcal{V}$. It is clear that each V_n is open. Indeed, if $x\in V_n$, then x has a neighborhood $W(x)$ which hits at most finitely many members of $\mathcal{G}_n$, and hence $W(x)\subset V_n$. It follows that $\mathcal{V}$ is an open collection. Now $\mathcal{G}_n$ is LF on V_n, and in particular, PF on V_n. By construction, it thus follows that $\mathcal{V}_n$ is PF for each $n\in N$, and the proof is complete.

Note that σ-*PF* open covers of countably metacompact spaces have *PF* open refinements and every subparacompact space is countably metacompact. Hence the following is now easy to show using Theorem 5.

Theorem 6. *For any strong-CWSN space X, the following are equivalent.*

(a) *X is subparacompact.* (b) *X is metacompact.*
(c) *X is θ-refinable.* (d) *X is $B(D,\omega)$-refinable.*

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