66. A Numerical Characterization of Ball Quotients for Normal Surfaces with Branch Loci

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We show that Kähler-Einstein geometry fits in with the theory of minimal models for normal surfaces with branch loci (cf. [14], [15]). One of important consequences of this is that an inequality of Miyaoka-Yau type holds for canonical normal surfaces with branch loci and with at worst log-canonical singularities and the equality characterizes ball quotients with finite volume. For details of this note, we refer to Sakai's survey article [15] on the classification of normal surfaces, Nakamura's master thesis [11] on the classification and uniformization of log-canonical surface singularities and Kobayashi's survey article [5] on uniformization of complex surfaces.

In this note, we mean by a divisor a Weil divisor, i.e., a linear combination of irreducible curves with integer coefficients. We use Mumford's intersection theory (see [10] and [15]) for $\boldsymbol{Q}$-divisors on normal surfaces. Let (V, D, p) be a pair of a germ of a normal complex surface V and a $\boldsymbol{Q}$-divisor $D=\sum_i(1-(1/b_i))D_i$ with $b_i=2, 3, \cdots, \infty$. We formally identify $\operatorname{Supp} D_i$ with a branch locus having the branch index b_i. In cace $b_i=\infty$, we consider the complement of such D_i. To understand such identification, it suffices to look at the coverings

(1) $$\boldsymbol{D} \ni z \longmapsto w=z^b \in \boldsymbol{D}$$

and

(2) $$\boldsymbol{D} \ni z \longmapsto w=\exp\left(\frac{z+1}{z-1}\right) \in \boldsymbol{D}^*,$$

where $\boldsymbol{D}$ and $\boldsymbol{D}^*$ denote the unit disk and the punctured unit disk, respectively. We say that a point p of V is a singularity of (V, D, p) if p is a singular point of V or if p is a smooth point of V and the curve $\operatorname{Supp}(D)$ has a singularity at p. Take a resolution

$$\mu:(\tilde{V}, \tilde{D}, \tilde{E}) \longrightarrow (V, D, p)$$

where $\tilde{D}$ and $\tilde{E}$ are the strict transforms of D and the exceptional set of $\mu: \tilde{V} \to V$, respectively. Let $\tilde{E}=\sum_\alpha E_\alpha$ be the decomposition of $\tilde{E}$ into irreducible components. Using Mumford's intersection theory, we define Δ by setting

(3) $$\mu^*(K_V+D)=K_{\tilde{V}}+\tilde{D}+\Delta.$$

(4) **Definition.** A singularity (V, D, p) is called *log-canonical* (resp. *log-terminal*) if there exists a good resolution μ such that $\Delta=\sum_\alpha a_\alpha E_\alpha$ with

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$a_\alpha \leq 1$ for all α (resp. $a_\alpha < 1$ for all α and $b_i < \infty$ for all i).

This definition does not depend on the choice of a good resolution. A log-canonical singularity is *LCS* if it is not log-terminal, i.e. some coefficients in $\tilde{D}+\Delta$ is equal to 1. Nakamura [11] classified log-canonical singularities and showed that all log-canonical singularities are quotient singularities in a broad sense, namely, they are uniformized by bounded symmetric domains:

(5) **Theorem** (Nakamura). *All log-canonical surface singularities are uniformized by bounded symmetric domains and are classified into the following four classes:*

(i) *Singularities of factor spaces $\boldsymbol{C}^2/\Gamma$ where Γ is a finite subgroup of $U(2)$ possibly containing reflections.*

(ii) *Singularities of the one point partial compactifications of factor spaces $H \times H/\Gamma$ where Γ is a parabolic discrete subgroup of* $\mathrm{Aut}(H \times H)$ *corresponding to a boundarg point, say, $(i\infty, i\infty)$, possibly containing reflections.*

(iii) *Singularities of the one point partical compactifications of factor spaces $\boldsymbol{B}^2/\Gamma$ where Γ is a parabolic discrete subgroup of* $\mathrm{Aut}(\boldsymbol{B}^2)$ *($\boldsymbol{B}^2$ is the open unit ball in $\boldsymbol{C}^2$) corresponding to a boundary point, say,* $(1, 0)$, *possibly containing reflections.*

(iv) *Singularities of the partial compactifications of factor spaces $\boldsymbol{D}^* \times \boldsymbol{D}^{(*)}/\Gamma$ where $\boldsymbol{D}^{(*)}$ stands for $\boldsymbol{D}$ or $\boldsymbol{D}^*$ and Γ is a finite subgroup of $U(2)$ possibly containing reflections operating linearly on $\boldsymbol{D}^* \times \boldsymbol{D}^{(*)} \subset \boldsymbol{C}^2$.*

(6) **Corollary.** *All log-canonical surface singularities are $\boldsymbol{Q}$-Gorenstein.*

Let (X, D) be a pair of a compact complex normal surface X and a $\boldsymbol{Q}$-divisor $D = \sum_i (1-(1/b_i))D_i$ $(b_i = 2, 3, \cdots, \infty)$ where D_i are irreducible curves on X. The log-canonical ring $\bar{R}(X, D)$ is defined to be the graded ring

$$\bigoplus_{m \geq 0} H^0(X, O(m(K_X + D))).$$

The log-Kodaira dimension $\bar{\kappa}(X, D)$ is the transcendence degree of $\bar{R}(X, D)$ (with the convention $\bar{\kappa}(X, D) = -\infty$ if $\bar{R}(X, D) = \boldsymbol{C}$).

(7) **Definition.** An irreducible curve C on X is a *log-exceptional curve of the first kind* (resp. *log-exceptional curve of the second kind*) if $C^2 < 0$ and $(K_X + D) \cdot C < 0$ (resp. if $C^2 < 0$ and $(K_X + D) \cdot C = 0$).

By successive contractions of log-exceptional curves of the first kind, we arrive at a *log-minimal model* (X', D') which by definition contains no log-exceptional curves of the first kind. If we further contract log-exceptional curves of the second kind in the log-minimal model, we arrive at a *log-canonical model* (X'', D''). We have the isomorphisms: $\bar{R}(X, D) \simeq \bar{R}(X', D') \simeq \bar{R}(X'', D'')$. As a consequence, we have $\bar{\kappa}(X, D) = \bar{\kappa}(X', D') = \bar{\kappa}(X'', D'')$. If $K_X + D$ is pseudoeffective, then the log-minimal model is unique and $K_{X'} + D'$ is numerically effective. If $\bar{\kappa}(X, D) = 2$, then the log-canonical model is unique and $K_{X''} + D''$ is numerically ample. Sakai [15] proved

(8) **Theorem.** *For a log-canonical model (X'', D'') with $\bar{\kappa}(X'', D'') = 2$,*

the log-canonical ring $\bar{R}(X'', D'')$ *is finitely generated if and only if* (X'', D'') *is* $\boldsymbol{Q}$*-Gorenstein. In this case,* $X''=\mathrm{Proj}(\bar{R}(X'', D''))$.

Although the $\boldsymbol{Q}$-Gorensteinness is not always preserved in the process of going to log-minimal and log-canonical models,***) it holds that if (X, D) has at worst log-canonical singularities, then so do its log-minimal and log-canonical models. Hence we have from (6) and (8)

(9) **Corollary.** *For* (X, D) *with at worst log-canonical singularities, the following conditions are equivalent*:

(i) $\bar{\kappa}(X, D)=2$,

(ii) $K_{X''}+D''$ *is numerically ample,*

(iii) $K_{X''}+D''$ *is an ample* $\boldsymbol{Q}$*-Cartier divisor.*

If one of the above conditions are fulfilled, then the log-canonical ring $\bar{R}(X, D)$ *is finitely generated and* $X''=\mathrm{Proj}\,\bar{R}(X, D)$.

Let (X, D) be as above. Suppose that (X, D) has at worst log-canonical singularities and that $\bar{\kappa}(X, D)=2$. Then (5) and (9) enable us to use the same strategy as in [3] to prove our main theorem.

(10) **Theorem.** *Let* (X, D) *be as above. Suppose that* (X, D) *has at worst log-canonical singularities and that* $\bar{\kappa}(X, D)=2$. *Then*

(i) $X_0''=X''-\bigcup_{b_i=\infty} D_i''-LCS(X'', D'')$ *with* $D_0''=D''\cap X_0''$ *is an orbifold with branch loci* $\mathrm{Supp}(D_0'')$ *with branch indices* $\{b_i\}$. *In particular, any singular point of* X'' *outside* $\bigcup_{b_i=\infty} D_i \cup LSC(X'', D'')$ *is an isolated quotient singularity and every end of* X_0'' *is uniformized by an end of a 2-dimensional locally Hermitian symmetric orbifold,*

(ii) *there exists a unique complete Kähler-Einstein orbifold-metric with negative scalar curvature on the orbifold* (X_0'', D_0'') *whose Kähler form* $\tilde{\omega}$ *defines a closed current on any resolution* μ: $Y''\to X''$ *and satisfies* $[\mu^*\tilde{\omega}]=2\pi c_1(\mu^*(K_{X''}+D''))$.

For a proof, we use (5) and (9) to construct a singular volume form Ψ with the following properties (cf. [3]):

(i) $\omega=-\mathrm{Ric}\,\Psi$ is a complete Kähler-orbifold metric on X_0'',

(ii) the metric ω is "asymptotically" equal to the canonical invariant metric at each end.

These conditions are achieved since the local uniformization theorem (5) gives us the canonical invariant metric in the level of Kähler potential at each end. We can then apply Cheng-Yau's method [1] (method of bounded geometry) to solve the Monge-Ampère equation

$$(\omega+i\partial\bar{\partial}u)^2=e^u\Psi \tag{11}$$

which derives a complete Kähler-Einstein orbifold metric $\tilde{\omega}=\omega+i\partial\bar{\partial}u$. The uniqueness in the cohomology class is a consequence of Yau's Schwarz lemma [18].

Integrating the Chern forms γ_1^2 and γ_2 of an orbifold-metric $\tilde{\omega}$ over X'' and applying the pointwise inequality $3\gamma_2-\gamma_1^2\geq 0$ valid for any Kähler-Einstein metric on a complex surface, we get the following inequality of

***) This is observed by Kawamata. See [15].

Miyaoka-Yau type:

(12) **Theorem.** *Let (X, D) and (X'', D'') be as in Theorem* (10). *Then we have*

$$(K_{X''}+D'')^2 \leq 3\left\{e(X''_0)+\sum_i\left(\frac{1}{b_i}-1\right)(e(D''_{0i})-d_i)+\sum_p\left(\frac{1}{|\Gamma(p)|}-1\right)\right\},$$

where $e(X''_0)$ means the Euler number of X''_0, etc., $D''_{0i}=D''_i \cap X''_0$, d_i is the number of singularities of (X'', D'') lying over D''_{0i}, and $|\Gamma(p)|$ is the order of the local fundamental group $\Gamma(p)$ of a log-terminal singular point p of (X'', D'') in the sense of orbifolds. The equality holds if and only if the orbifold (X''_0, D''_0) is biholomorphic to a ball quotient $\boldsymbol{B}^2/\Gamma$ with Γ a discrete subgroup of $\mathrm{Aut}(\boldsymbol{B}^2)$. *In this case, $\bigcup_{b_i\neq\infty}\mathrm{Supp}(D''_0)$ are the branch loci with branch indices $\{b_i\}$.*

Our inequality generalizes Miyaoka-Yau inequalities so far obtained for complex surfaces (see [2], [3], [4], [6], [7], [13], [19] and [20]) and is best possible in the sense that only log-canonical singularities with branch loci appear in compactified ball quotients. It would be interesting to study the converse of Theorem (10). Recently, Mok [8] and Mok-Zhong [9] solved the problem of characterizing such compactifiable complex (2-dimensional) orbifolds (X''_0, D''_0) as in Theorem (10) as (topologically finite) complete Kähler (-Einstein) orbifolds with negative Ricci-curvature with a finite volume. See also Siu-Yau [16], Nadel-Tsuji [12] and Tsuji [17] on compactifications of (negatively curved) complete Kähler (-Einstein) manifolds with a finite volume.

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