63. *On the Unitarizability of Principal Series Representations of $\mathfrak{p}$-adic Chevalley Groups*

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1. In this note, we shall determine the unitarizability of unramified principal series representations of $\mathfrak{p}$-adic Chevalley groups of classical types. Detailed proofs of all the results stated here are given in [7].

2. Let k be a non-archimedean local field, $\mathfrak{D}$ be the maximal compact subring and ϖ be a prime element of k. Set $q=|\mathfrak{D}/\varpi\mathfrak{D}|$. The following theorem is our main tool in this research.

Theorem 1. *Let N be the group of k-rational points of a unipotent algebraic group defined over k. Let T be a distribution of positive type on N. Then, for any $\alpha \in C_c^\infty(N)$, the convolution $T*\alpha$ is a bounded function on N.*

3. Let $\boldsymbol{G}$ be a universal Chevalley group defined over k in the sense of Steinberg [6]. Let $\boldsymbol{T}$ be a maximal k-split torus and $\boldsymbol{B}$ be a Borel subgroup defined over k which contains $\boldsymbol{T}$. Let $\boldsymbol{N}$ be the unipotent radical of $\boldsymbol{B}$. Let G, T, B and N stand for the groups of k-rational points of $\boldsymbol{G}$, $\boldsymbol{T}$, $\boldsymbol{B}$ and $\boldsymbol{N}$ respectively. Let Σ be the root system and $\Delta=\{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ be the set of simple roots determined by $(\boldsymbol{G}, \boldsymbol{B}, \boldsymbol{T})$, where ℓ is the rank of $\boldsymbol{G}$. Let Σ^+ be the set of positive roots and W be the Weyl group. For $w \in W$, set $\Psi_w^+=\{\alpha\in\Sigma^+ \mid w\alpha<0\}$. We have $B=TN=NT$ and T (resp. N) is generated by $h_\alpha(t)$ (resp. $x_\alpha(t)$) for $\alpha\in\Sigma^+$, $t\in k^\times$ (resp. $t \in k$) in the notation of [6]. If $\alpha\in\Sigma$, let $\check{\alpha}\in \mathrm{Hom}(\boldsymbol{G}_m, \boldsymbol{T})$ be the co-root of α and set $a_\alpha=\check{\alpha}(\varpi)=h_\alpha(\varpi)\in T$. For α, $\beta\in\Sigma$, we set $\langle\alpha,\beta\rangle=\langle\alpha,\check{\beta}\rangle_1$ with the canonical pairing $\langle\ ,\ \rangle_1$ of a root with a co-root. Let δ_B denote the modular function of B. For a quasi-character χ of T, let $PS(\chi)$ denote the space of all locally constant functions φ on G which satisfy

$$\varphi(tng)=\delta_B(t)^{1/2}\chi(t)\varphi(g) \qquad \text{for any } t\in T,\ n\in N,\ g\in G.$$

Let $\pi(\chi)$ denote the admissible representation of G realized on $PS(\chi)$ by right translations.

Let K be the subgroup of G generated by $x_\alpha(t)$, $\alpha\in\Sigma$, $t\in\mathfrak{D}$. Then K is a maximal compact subgroup of G and we have the Iwasawa decomposition $G=BK$. We call χ *unramified* if χ is trivial on $T\cap K$, the group generated by $h_\alpha(t)$, $\alpha\in\Sigma^+$, $t\in\mathfrak{D}^\times$. Let X be the group of all unramified quasi-characters of T. The map $\chi\to(\chi(a_{\alpha_1}), \chi(a_{\alpha_2}), \cdots, \chi(a_{\alpha_\ell}))$ defines an isomorphism $X\cong(\boldsymbol{C}^\times)^\ell$ and we consider X as a complex Lie group. We call χ *regular* if $w\chi\neq\chi$ for any $w\in W$, $w\neq 1$. Let X^r (resp. X^i) denote the set of all $\chi\in X$ which are regular (resp. regular and $\pi(\chi)$ is irreducible). Let

$w \in W$. We set $X_w = \{\chi \in X \mid w\chi = \bar{\chi}^{-1}\}$, $X_w^r = X_w \cap X^r$, $X_w^i = X_w \cap X^i$. Taking $x_w \in K$ which represents w, we define an intertwining operator T_w from $PS(\chi)$ to $PS(w\chi)$ by

$$(T_w(\varphi))(g) = \int_{wNw^{-1} \cap N \backslash N} \varphi(x_w^{-1} n g) dn, \quad \varphi \in PS(\chi), \quad g \in G,$$

with the invariant measure dn suitably normalized. This integral is absolutely convergent if $|\chi(a_\alpha)| < 1$ for any $\alpha \in \Psi_w^+$ and can be meromorphically continued to the whole X; T_w is holomorphic at χ if $\chi(a_\alpha) \neq 1$ for any $\alpha \in \Psi_w^+$. In particular, T_w is holomorphic on X^r.

4. We assume $\chi \in X^i$ until the end of 5. If $\pi(\chi)$ is hermitian, there exists a unique $w \in W$ such that $\chi \in X_w^i$, $w^2 = 1$. Then $\pi(\chi)$ is unitarizable if and only if the Hermitian form

$$(1) \qquad (\varphi_1, \varphi_2) = c \int_{B \backslash G} (T_w(\varphi_1))(g) \overline{\varphi_2(g)} dg, \qquad \varphi_1, \varphi_2 \in PS(\chi)$$

is positive definite with $c = \pm 1$. Let w_0 be the longest element of W and ω_0 be an element of K which represents w_0. Since Bw_0N is the big cell, we see easily that for every $\Phi \in C_c^\infty(N)$, the exists a unique $\varphi \in PS(\chi)$ such that $\Phi(n) = \varphi(\omega_0 n)$, $n \in N$. We put $\varphi = \iota_\chi(\Phi)$. Then

$$(2) \qquad T_\chi(\Phi) = T_w(\iota_\chi(\Phi))(\omega_0), \qquad \Phi \in C_c^\infty(N)$$

defines a distribution on N. By (1), we have

$$(\varphi_1, \varphi_2) = c \int_N (T_w(\varphi_1))(\omega_0 n) \overline{\varphi_2(\omega_0 n)} dn, \qquad \varphi_1, \varphi_2 \in PS(\chi),$$

and this formula shows that cT_χ is of positive type if $\pi(\chi)$ is unitarizable.

For a subset J of Δ, let W_J denote the group generated by the reflexions obtained from J and let w_J be the longest element of W_J. It is known (cf. [2], p. 225) that any element of order 2 of W is conjugate to w_J for some $J \subseteq \Delta$. Since $\pi(w_1\chi) \cong \pi(\chi)$ for any $w_1 \in W$, we may assume $\chi \in X_{w_J}^i$ for some $J \subseteq \Delta$ without losing any generality. Let Σ_J be the root system generated by J and set

$$\Sigma_J^+ = \Sigma_J \cap \Sigma^+, \quad n_J(\alpha) = \sum_{\beta \in \Sigma_J^+} \langle \beta, \alpha \rangle \quad \text{for } \alpha \in \Sigma_J.$$

By Theorem 1, we see that $T_\chi * f$ is bounded on N for any $f \in C_c^\infty(N)$ if $\pi(\chi)$ is unitarizable. We choose f as the characteristic function of U_1^+, the subgroup of $N \cap K$ generated by $x_\alpha(t)$, $\alpha \in \Sigma^+$, $t \in \varpi\mathfrak{D}$. Then we obtain

Theorem 2. *Let $\chi \in X_{w_J}^i$ and assume that $\pi(\chi)$ is unitarizable. Then we have*

$$q^{-n_J(\alpha)/2} < |\chi(a_\alpha)| < q^{n_J(\alpha)/2} \qquad \text{for any } \alpha \in \Sigma_J^+.$$

Corollary 3. *If w_J acts as the multiplication by -1 on J, then we have*

$$(3) \qquad q^{-1} < |\chi(a_\alpha)| < q \qquad \text{for any } \alpha \in \Sigma_J.$$

If $\chi \in X^r$, then $\pi(\chi)$ has the unique irreducible quotient (cf. [1], p. 304), which we denote by π_χ. In the similar way as above, we obtain

Proposition 4. *If $\chi \in X_{w_J}^r$ and π_χ is unitarizable, then we have*

$$q^{-n_J(\alpha)/2} \leq |\chi(a_\alpha)| \qquad \text{for any } \alpha \in \Sigma_J^+.$$

5. We combine Corollary 3 with certain deformation arguments on representations.

Proposition 5. *Let w, w_1, $w_2 \in W$ be elements of order 2 such that $w = w_1 w_2$, $l(w) = l(w_1) + l(w_2)$, where l denotes the length. Let p: $[0,1] \to X_w$ and p_1: $[0,1] \to X_{w_1}$ be continuous maps. Put $\chi_t = p(t)$, $\chi_t^1 = p_1(t)$ for $0 \leq t \leq 1$. We assume the following conditions.*

(i) $\chi_0 = \chi_0^1$.

(ii) $p(0,1] \subseteq X_w^i$ *and* $p_1(0,1] \subseteq X_{w_1}^i$.

(iii) *For any* $\alpha \in \Psi_{w_1}^+$, $\chi_0(a_\alpha) \neq 1$, q.

(iv) *For any* $\alpha \in \Psi_{w_2}^+$, $\chi_0(a_\alpha) = 1$.

Then $\pi(\chi_{t_0}^1)$ (resp. $\pi(\chi_{t_0})$) *is unitarizable for some* $t_0 \in (0,1]$ *if and only if* $\pi(\chi_t)$ (resp. $\pi(\chi_t^1)$) *is unitarizable for* $0 < t \leq 1$.

We consider the cases of types B, C and D separately (we omit the discussion for type A). We realize Σ as in "Planches" of Bourbaki [2]. Without losing any generality, we may normalize J in the following forms. If Σ is of type B_ℓ or C_ℓ, $J = \{\alpha_1, \alpha_3, \cdots, \alpha_{2m-1}, \alpha_n, \alpha_{n+1}, \cdots, \alpha_{\ell-1}, \alpha_\ell\}$, $2m < n$. We put $n = \ell + 1$ if $\alpha_\ell \notin J$. If Σ is of type D_ℓ, $J = \{\alpha_1, \alpha_3, \cdots, \alpha_{2m-1}\} \cup J_1$, where $J_1 = \{\alpha_n, \alpha_{n+1}, \cdots, \alpha_{\ell-1}, \alpha_\ell\}$, $2m < n$, $|J_1| \geq 4$ and even, or $J_1 = \emptyset$, $2m \leq \ell - 1$ or $J_1 \subseteq \{\alpha_{\ell-1}, \alpha_\ell\}$, $2m < \ell - 1$.

Under these normalizations, w_J acts as -1 on J. Hence (3) is a necessary condition for the unitarizability.

Theorem 6. *Assume **G** is of type Y_ℓ and let $\chi \in X_{w_J}^i$, where $Y = B$, C or D. Then $\pi(\chi)$ is unitarizable if and only if the conditions* (3) *and* (Y) *are satisfied. Here*

(B) $\chi(a_{\alpha_\ell}) > 0$ *if* $\alpha_\ell \in J$, $\chi(a_{\alpha_{2m-1}}) > 0$ *if* $\alpha_\ell \notin J$.

(C) *The number of indices i such that $\chi(a_{2\varepsilon_i}) < 0$, $n \leq i \leq \ell$, is even.*

(D) $\chi(a_\alpha) > 0$ *for any* $\alpha \in J_1$.

We can prove this theorem by induction on $|J|$ applying Proposition 5 and its variants.

6. Let $\chi \in X$. Then $\pi(\chi)$ is of finite length and has a unique spherical constituent π_χ^1 (cf. [3]). Let $\boldsymbol{P}$ be the set of all $\chi \in X$ such that π_χ^1 is unitarizable. Then $\boldsymbol{P}$ is a compact subset of X which is stable under W.

Theorem 7. *Assume **G** is of classical type and let $\chi \in X$. If $\pi(\chi)$ is irreducible and unitarizable, then χ belongs to the closure of $\boldsymbol{P} \cap X^i$.*

Since we have determined $\boldsymbol{P} \cap X^i$ explicitly by Theorem 6, this completes the determination of unitarizability of $\pi(\chi)$, $\chi \in X$, when $\pi(\chi)$ is irreducible.

References

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