7. *Iwasawa Theoretical Residue Formulas for Algebraic Tori**)

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Introduction. In his paper [2], J. Coates gave an Iwasawa theoretical analogue of the analytic class number formula. The aim of this note is to give a residue formula for an algebraic torus, which generalizes Coates' formula under a certain condition and can also be regarded as an Iwasawa theoretical analogue of the analytic class number formula for an algebraic torus introduced by T. Ono and J.-M. Shyr ([4], [10]). We must mention that Iwasawa theory for algebraic tori was developed by P. Schneider, and both our result and proof were suggested by his papers ([6], [7]).

Finally I would like to express my sincere gratitude to Prof. Takashi Ono for his warmhearted encouragement and dedicate this paper to him.

§ 1. Iwasawa *L*-function for an algebraic torus. In this section, we shall prepare the notations and assumptions which will be necessary below and define an Iwasawa L-function associated to the character module of an algebraic torus.

p : an odd prime number. μ_{p^n} : the group of p^n-th roots of unity. $\mu_{p^\infty}:=\bigcup_{n\geq 1}\mu_{p^n}$. F : a totally real finite algebraic number field. T : an algebraic torus defined over F. $\hat{T}$: the group of rational characters of T. K : the minimal Galois splitting field of T. In the following, we assume the next condition (a).

(a) K is also totally real and p is unramified in $K/\boldsymbol{Q}$.

S : the set of primes of F which lie over p, ∞, or ramify in K/F. F_S : the maximal extension over F unramified outside S. For simplicity, we put $\widehat{T(p)}:=\hat{T}\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_p/\boldsymbol{Z}_p$ and denote by H_S and G the Galois groups $G(F_S/F(\mu_{p^\infty}))$ and $G(F(\mu_{p^\infty})/F)$ respectively. We shall use standard notations in Galois cohomology theory (e.g. [8]). Now, the Galois group G acts continuously on the Galois cohomology groups $H^i(H_S, \widehat{T(p)})$ $(i\geq 0)$. For the Galois group G, we have the canonical decomposition $G=\Delta\times\Gamma$, $\Delta:=G(F(\mu_p)/F)$, $\Gamma:=G(F(\mu_{p^\infty})/F(\mu_p))\cong\boldsymbol{Z}_p$. If we denote by $H_i(T)$ the Δ-invariants of the Pontrjagin duals of $H^i(H_S, \widehat{T(p)})$, $H_i(T)$ are compact modules over completed group ring $\boldsymbol{Z}_p[[\Gamma]]$. Concerning the structures of $\boldsymbol{Z}_p[[\Gamma]]$-modules $H_i(T)$, we can prove the following.

*) This paper is a part of my master's thesis in Tokyo Metropolitan University at 1988.

(0) $H_0(T)$ is a finitely generated Z_p-module of rank $\rho_T := \mathrm{rank}_Z \hat{T}(F)$, $\hat{T}(F)$:= the group of rational characters of T defined over F, and Γ acts trivially on $H_0(T)$.
(1) $H_1(T)$ is a finitely generated torsion $Z_p[[\Gamma]]$-module.
(2) $H_i(T)=0$ if $i \geq 2$.
It is easy to see (0) by the definition. To prove (1), we use the well known fact from Iwasawa theory (e.g. [3], p. 94) and a little cohomological consideration to reduce to the 'G_m-case'. Since the cohomological p-dimension $\mathrm{cd}_p(H_S) \leq 2$, $H^i(H_S, \widehat{T(p)})=0$ for $i>2$ and $H^2(H_S, \widehat{T(p)})$ is divisible. Since $H^2(F_S/K(\mu_{p^\infty}), Q_p/Z_p)=0$ ('Weak Leopoldt's conjecture' e.g. [11], p. 348), (2) follows from the spectral sequence: $H^i(K(\mu_{p^\infty})/F(\mu_{p^\infty}), H^j(F_S/K(\mu_{p^\infty}), \widehat{T(p)})) \Rightarrow H^{i+j}(H_S, \widehat{T(p)})$.

By the above, the following definition makes sense.

$$f_i(x) := p^{\mu_i(T)} \det (x-(\gamma-1) \mid H_i(T) \otimes_{Z_p} Q_p) \in Q_p[x] \qquad (i \geq 0)$$

where $\gamma := \kappa^{-1}(1+p^e)$ for the p-cyclotomic character $\kappa: \Gamma \xrightarrow{\sim} 1+p^e Z_p$ and $\mu_i(T)$:= the Iwasawa μ-invariant of $Z_p[[\Gamma]]$-module $H_i(T)$.

Definition of Iwasawa L-function for T.

$$L(T; s) := \prod_{i \geq 0} f_i((1+p^e)^{1-s}-1)^{(-1)^i} \qquad (s \in Z_p)$$

Remark. If $T=G_{m/F}$, the above L-function is nothing but the Iwasawa zeta function of F in Coates' formula up to a p-adic unit.

Let $\mathfrak{p}$ be any prime of F over p, $F_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of F and $O_\mathfrak{p}$ be the ring of integers in $F_\mathfrak{p}$. Let $T(O_\mathfrak{p})$ be the maximal compact subgroup of $T(F_\mathfrak{p})$, the group of $F_\mathfrak{p}$-rational points in T. Then we assume also the following.

(b) The $\mathfrak{p}$-adic logarithm $\log_\mathfrak{p}$ on the $\mathfrak{p}$-adic Lie group $T(O_\mathfrak{p})$ ([1], III 7.6) induces the isomorphism: $p^{-1}\log_\mathfrak{p}: T(O_\mathfrak{p})/\text{torsion} \xrightarrow{\sim} O_\mathfrak{p}^g$, $g := \dim T$.

Remark. Under the assumption (a) (p is unramified in K/Q), the above assumption (b) is satisfied for $T=R_{E/F}(G_m)$, where E is any intermediate field of K/F. Thus there are many examples of tori which satisfy (a), (b) and whose Tamagawa numbers are not zero.

§2. Iwasawa theoretical residue formula for an algebraic torus. Besides the assumptions (a), (b) in §1, we assume that the Leopoldt conjecture for (K, p) holds. Our formula is the following.

Theorem. *Under the assumptions as above, $L(T; s)$ has a pole of order $\rho_T := \mathrm{rank}_Z \hat{T}(F)$ at $s=1$, and its residue has the same p-adic valuation as*

$$\frac{h_T R_p(T)}{\tau(T) c(T)} \prod_{\mathfrak{p} \in S \setminus \{\infty\}} \sharp(T(p)(F_\mathfrak{p})) \prod_{\mathfrak{p} \mid p} \frac{\sharp(T^{(\mathfrak{p})}(\kappa_\mathfrak{p}))}{(N\mathfrak{p})^g}.$$

We must explain the notations in the above theorem.

$\sharp$: the cardinality. By p-adic valuation, we mean the normalized p-adic absolute value of C_p(:= the p-adic completion of the fixed algebraic closure of Q_p). h_T the class number of T. $T(p)$: the group of all p-power torsion points of T. $T(p)(F_{\mathfrak{p}}):=T(p)\cap T(F_{\mathfrak{p}})$. $\kappa_{\mathfrak{p}}$: the residue field of F at $\mathfrak{p}$. $T^{(\mathfrak{p})}$: the reduction of T at $\mathfrak{p}$. $T^{(\mathfrak{p})}(\kappa_{\mathfrak{p}})$: the group of $\kappa_{\mathfrak{p}}$-rational points in $T^{(\mathfrak{p})}$. N: the absolute norm. Next, let us define $R_p(T)$, the p-adic regulator of T. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be the primes of F over p. We denote by $\log_{i,j}$ $(1\leq j\leq g)$ the j-th component of $\log_{\mathfrak{p}_i}$ in § 1. For $\mathfrak{p}_i | p$, let $\phi_i^1, \cdots, \phi_i^{d_i}: F_{\mathfrak{p}_i} \xrightarrow{\subset} \boldsymbol{C}_p$ be all embeddings of $F_{\mathfrak{p}_i}$ into $\boldsymbol{C}_p$ over $\boldsymbol{Q}_p$, where $d_i:=[F_{\mathfrak{p}_i}:\boldsymbol{Q}_p]$. We put $\log_{i,j}^k:=\phi_i^k\circ\log_{i,j}$. Since the unit group $T(O_F)$ is a finitely generated abelian group of rank $t:=g[F:\boldsymbol{Q}]-\rho_T$ according to Shyr [9], we denote by $\varepsilon_1, \cdots, \varepsilon_t$ the basis of $T(O_F)/\text{torsion}$ and define $R(\varepsilon_1, \cdots, \varepsilon_t)$ by the following $t\times g[F:\boldsymbol{Q}]$-matrix.

$$R(\varepsilon_1, \cdots, \varepsilon_t):=\begin{pmatrix} \log_{1,1}^1(\varepsilon_1), \cdots, \log_{1,1}^{d_1}(\varepsilon_1), \log_{1,2}^1(\varepsilon_1), & \cdots, \log_{r,g}^1(\varepsilon_1), \cdots, \log_{r,g}^{d_r}(\varepsilon_1) \\ \cdots\cdots & \cdots\cdots \\ \log_{1,1}^1(\varepsilon_t), \cdots, \log_{1,1}^{d_1}(\varepsilon_t), \log_{1,2}^1(\varepsilon_t), & \cdots, \log_{r,g}^1(\varepsilon_t), \cdots, \log_{r,g}^{d_r}(\varepsilon_t) \end{pmatrix}.$$

As we can prove under Leopoldt's conjecture that the rank of the matrix $R(\varepsilon_1, \cdots, \varepsilon_t)$ is equal to t, we define a p-adic regulator $R_p(T)$ of T as follows.

$$R_p(T):=\min\{|\det R|_p^{-1} \mid R=r\times r\text{-submatrix of } R(\varepsilon_1, \cdots, \varepsilon_t) \text{ and } \det R\neq 0\},$$

where $|\cdot|_p:=$the p-adic absolute value. $(|p|_p^{-1}=p)$.

We remark that $R_p(T)$ is independent of the choice of $\{\varepsilon_i\}$ and coincides with the Leopoldt's p-adic regulator of F if $T=\boldsymbol{G}_{m/F}$.

Finally let us define $c(T)$. We always denote by $A(p)$ the p-power torsion part of an abelian group A. If we put $\mathrm{Ker}(F, T(p)):=\mathrm{Ker}(H^1(F, T(p))\to\prod_{\mathfrak{p}:\text{all}} H^1(F_{\mathfrak{p}}, T(p)))$ and $\text{Ш}_F(T):=\mathrm{Ker}(H^1(F, T)\to\prod_{\mathfrak{p}:\text{all}} H^1(F_{\mathfrak{p}}, T))$, we have the natural homomorphism $\varphi: \mathrm{Ker}(F, T(p))\to\text{Ш}_F(T)(p)$. We put $c_1(T):=\#\mathrm{coker}(\varphi)$. ($\text{Ш}_F(T)$ is finite. [5]). Let $T(A_F)$ and $\mathrm{Cl}_F(T)$ be the adele group and the class group of T respectively. Then we have the natural homomorphism $\psi: (T(A_F)/\overline{T(F)\prod_{\mathfrak{p}|\infty} T(F_{\mathfrak{p}})\prod_{\mathfrak{p}\notin S} T(O_{\mathfrak{p}})})(p)\to\mathrm{Cl}_F(T)(p)$, where —means the topological closure with respect to the adele topology. We put $c_2(T):=\#\mathrm{coker}(\psi)\cdot$($\mathrm{Cl}_F(T)$ is finite. [4]). Let us give another description of $c_2(T)$. If we put $\mathrm{Ker}_F(T):=\mathrm{Ker}(T(F)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_p/\boldsymbol{Z}_p\to\prod_{\mathfrak{p}:\text{all}} T(F_{\mathfrak{p}})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_p/\boldsymbol{Z}_p)$ we have the natural homomorphism as follows. For $x\otimes p^{-i}$ in $\mathrm{Ker}_F(T)$, there exists $x_{\mathfrak{p}}$ in $T(F_{\mathfrak{p}})$ such that $x\equiv x_{\mathfrak{p}}^{p^i} \bmod T(O_{\mathfrak{p}})$ for any $\mathfrak{p}\nmid\infty$. We can define a homomorphism $\psi': \mathrm{Ker}_F(T)\to\mathrm{Cl}_F(T)(p)$ by $\psi'(x\otimes p^{-i}):=$the class of $(x_{\mathfrak{p}})_{\mathfrak{p}}$ in $\mathrm{Cl}_F(T)$. Then we can show $\mathrm{image}(\psi)=\mathrm{image}(\psi')$. We define $c(T):=c_1(T)c_2(T)$.

Remark. (1) Concerning $c_2(T)$, by using class field theory, we can prove the following under our assumptions.

If $T=R_{E/F}(\boldsymbol{G}_m)$, where E is any intermediate field of K/F, ψ is surjective, i.e., $c_2(T)=1$.

(2) If $T=\boldsymbol{G}_{m/F}$, we have $c_1(T)=c_2(T)=1$. Therefore our formula is reduced to Coates' formula under the assumption that p is unramified in $F/\boldsymbol{Q}$.

§ 3. Outline of the proof of Theorem. We shall use the notations introduced in § 1 and § 2.

By (0) in §1, $f_0((1+p^e)^{1-s}-1)=((1+p^e)^{1-s}-1)^{\rho_T}$. This implies that $L(T;s)$ has a pole of order ρ_T at $s=1$ since $f_1(0)\neq 0$ as we shall see below. First, we have $H^0(\Gamma, H_1(T))=H^1(G, H^1(H_S, \widehat{T(p)}))^*=0$ (* means the Pontrjagin dual) by considering the spectral sequence $H^i(G, H^j(H_S, \widehat{T(p)}))$ $\Rightarrow H^{i+j}(F_S/F, \widehat{T(p)})$, $\mathrm{cd}_p(G)=1$ and $H^2(F_S/F, \widehat{T(p)})=0$. Here $H^2(F_S/F, \widehat{T(p)})$ $=0$ follows from the Leopoldt conjecture for (K, p). Therefore we have $|f_1(0)|_p=\#(H^1(\Gamma, H_1(T)))^{-1}$ by the definition of $f_1(x)$ and moreover we can easily see $\#(H^1(\Gamma, H_1(T)))=\#(H^0(G, H^1(H_S, \widehat{T(p)})))=\#(H^1(F_S/F, \widehat{T(p)})_{\mathrm{div}})$ $\#(H^1(F, \hat{T})(p))^{-1}$, where A_{div} means the quotient of A by the maximal divisible subgroup. By Tate-Poitou exact sequence, we have $\#(H^1(F_S/F, \widehat{T(p)})_{\mathrm{div}})$ $=\#(\mathrm{Ker}\,(F, T(p)))\prod_{\mathfrak{p}\in S\setminus\{\infty\}}\#(T(p)(F_{\mathfrak{p}}))$. The finiteness of $\mathrm{Ker}\,(F, T(p))$ also follows from the Leopoldt conjecture for (K, p). If we put $\mathrm{Ker}_{O_F}(T) :=$ $\mathrm{Ker}\,(T(O_F)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_p/\boldsymbol{Z}_p\to\prod_{\mathfrak{p}|p}T(O_{\mathfrak{p}})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_p/\boldsymbol{Z}_p)$, we can see the kernels of φ and ψ' are $\mathrm{Ker}_F(T)$ and $\mathrm{Ker}_{O_F}(T)$ respectively. Here we remark that the Tamagawa number $\tau(T)$ is equal to $\#(H^1(F, \hat{T}))\,\#(\text{Ш}_F(T))^{-1}$ by [5]. Therefore, if we can calculate $\#(\mathrm{Ker}_{O_F}(T))$, all is done. However, we can see $\#(\mathrm{Ker}_{O_F}(T))=\#((\prod_{\mathfrak{p}|p}T(O_{\mathfrak{p}})/\mathrm{torsion})/(T(O_F)/\mathrm{torsion}))(p))$. So, taking $\mathfrak{p}$-adic logarithm, this index can be calculated as $p^{-t}R_p(T)$ by elementary divisor theory.

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