53. *Loop Groups and Related Affine Lie Algebras*

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Introduction. We are concerned with the Lie group $\tilde{G}_k$ of C^k-loops in a connected, simply connected complex simple Lie group G, its Lie algebra $\tilde{\mathfrak{g}}_k$, and a central extension $\hat{\mathfrak{g}}_k$ of $\tilde{\mathfrak{g}}_k$. The Lie algebra $\tilde{\mathfrak{g}}$ of algebraic loops in the Lie algebra $\mathfrak{g}$ of G has a universal central extension $\hat{\mathfrak{g}}$ called an affine Lie algebra, and the corresponding 2-cocycle $Z(\cdot,\cdot)$ was explicitly given in [1]. We extend the 2-cocycle of $\tilde{\mathfrak{g}}_k$ after [2], and get a central extension $\hat{\mathfrak{g}}_k$ of $\tilde{\mathfrak{g}}_k$. $\hat{\mathfrak{g}}$ is one of the simplest infinite-dimensional Kac-Moody algebras. The corresponding Kac-Moody group $\hat{G}$ is a 1-dimensional central extension of the group $\tilde{G}$ of algebraic loops in G (cf. [1], [7], [4], and [5]).

Since the kernel of the adjoint action Ad of $\hat{G}$ on $\hat{\mathfrak{g}}$, is precisely the center C of $\hat{G}$, $\tilde{G}\simeq\hat{G}/C$ acts on $\hat{\mathfrak{g}}$ through Ad, and the set of invariants in $\hat{\mathfrak{g}}$ under this action is just the center of $\hat{\mathfrak{g}}$. The action on $\hat{\mathfrak{g}}$ induces the adjoint action of $\tilde{G}$ on $\tilde{\mathfrak{g}}$. The main purpose of this article is to construct a completed version of this fact for the pair of the infinite-dimensional Lie group $\tilde{G}_k$ and the Lie algebra $\hat{\mathfrak{g}}_k$.

§ 1. The coefficient extension from C to $L_k=C^k(S^1)$. Let $L_k:=C^k(S^1)$, the algebra of C^k-functions on S^1. This becomes a Banach algebra if we introduce a norm $|\cdot|_k$ as

$$|a|_k:=\sup_{r\in\boldsymbol{R},\, j=0,\cdots,k}|(\partial^j a)(e^{2\pi\sqrt{-1}r})|\qquad \text{for } a\in L_k,$$

where ∂ is a differential operator on S^1, defined by

$$(\partial a)(e^{2\pi\sqrt{-1}r}):=\frac{1}{2\pi\sqrt{-1}}\frac{d}{dr}a(e^{2\pi\sqrt{-1}r}).$$

Let n be a positive integer and $i=0,1,2,\cdots,n$. Define derivations D_i on the polynomial ring $P_{k;n}:=L_k[X_1,\cdots,X_n]$ by $D_iX_{i'}=\delta_{ii'}$, and $D_ia=0$ for $i'=1,2,\cdots,n$, $a\in L_k$. For a bounded closed subset B in $(L_k)^n$ and a non-negative integer j, put

$$|f|_{k;B,j}:=\sup_{m,b}|(D^m f)(b)|_k\qquad \text{for } f\in P_{k;n},$$

where sup is taken over all $m=(m_1,\cdots,m_n)\in(\boldsymbol{Z}_{\geqq 0})^n$ satisfying $|m|:=m_1+\cdots+m_n\leqq j$, and all $b=(b_1,\cdots,b_n)\in B$, and D^m means $D_1^{m_1}D_2^{m_2}\cdots D_n^{m_n}$. Let $C^{k;j}(B)$ be the completion of the normed space $(P_{k;n},|\cdot|_{k;B,j})$.

Let U be an open set in $(L_k)^n$, and $C^{k;j}(U)$ the space of maps $f:U\to L_k$ which satisfy that, for any $u\in U$, there exist a bounded closed neighbourhood B of u in $(L_k)^n$ and $g\in C^{k;j}(B)$ such that $f(b)=g(b)$ for $\forall b\in B$. We define $D^m f(u)$ as $D^m g(u)$ for $m\in(\boldsymbol{Z}_{\geqq 0})^n$, $|m|\leqq j$.

Proposition 1.1. *Every $f \in C^{k;j}(U)$ is C^j-map of U into L_k.*

Let V be an open set in $\boldsymbol{C}^n$. $V(L_k)$ is open in $(L_k)^n$. For $f \in C^j(V)$, we define a may $\Psi(f)$ of $V(L_k)$ into L_k by $\Psi(f)(a_1, \cdots, a_n)(t) := f(a_1(t), \cdots, a_n(t))$ for $(a_1, \cdots, a_n) \in V(L_k), t \in S^1$.

Proposition 1.2. *Ψ maps $C^{k+j}(V)$ into $C^{k;j}(V(L_k))$.*

For an open subset U of $(L_k)^n$, and $m \in \boldsymbol{Z}_{>0}$, put

$$C^{k;j}(U; (L_k)^m) := C^{k;j}(U) \times \cdots \times C^{k;j}(U) \quad (m \text{ times}).$$

The following proposition is clear from Proposition 1.1.

Proposition 1.3. *Every element in $C^{k;j}(U; (L_k)^m)$ is a C^j-map from U into $(L_k)^m$.*

For every element $f = (f_1, \cdots, f_n)$ in $C^{k+j}(V; \boldsymbol{C}^m)$ (=the space of C^{k+j}-maps from V into $\boldsymbol{C}^m$), we put

$$\Psi(f) := (\Psi(f_1), \cdots, \Psi(f_n)).$$

The same fact as Proposition 1.2 holds for this new Ψ as follows.

Proposition 1.4. *For any $f \in C^{j+k}(V; C^m)$, $\Psi(f)$ belongs to $C^{k;j}(V(L_k); (L_k)^m)$.*

Let $L_{k,\boldsymbol{R}}$ be the real form of L_k consisting of the real valued functions. All the above results are also true for $L_{k,\boldsymbol{R}}$.

§ 2. Structure of the space of loops in a manifold. Let $k = 0, 1, 2, \cdots$, and M be an n-dimensional C^k-manifold. We denote by $M(L_k)$ the space of C^k-loops in M:

$$M(L_k) = \{f : S^1 \longrightarrow M; f \text{ is of class } C^k\}.$$

For another finite-dimensional manifold M', and a C^k-map F from M into M', we define a map $\Psi(L_k)$ into $M'(L_k)$ by

$$\Psi(F)(f)(s) := F(f(s)) \qquad \text{for } f \in M(L_k),\ s \in S^1.$$

Proposition 2.1. *There exists a topology on $M(L_k)$, with respect to which the above $\Psi(F)$'s are continuous, and the decomposition of $M(L_k)$ into the connected components with this topology, is exactly given by the homotopy classes of M.*

Now, we consider the case of a Lie group M. Thanks to Propositions 2.1 and 1.4, $M(L_k)$ becomes a Lie group with tangent space $(L_k)^{\dim M}$ as follows.

Theorem 2.3. *Let M be a finite-dimensional Lie group, and $k = 0, 1, 2, \cdots, \infty$. For any coordinate neighbourhood (V, ϕ) of 1 in M, there exists a unique Lie group structure on $M(L_k)$ in which a coordinate neighbourhood of 1 is given by $(V(L_k), \Psi(\phi))$. This structure is independent from the choice of (V, ϕ). For another Lie group M' of finite dimension, any open subset U of M, and a C^∞-map F of U into M', the map $\Psi(F)$* of *$U(L_k)$ into $M'(L_k)$ is of class C^∞.*

§ 3. A completed affine Lie algebra. Let H be a Cartan subgroup of G, and $\mathfrak{h}$ its Lie algebra. Denote by Δ the root system of $(\mathfrak{g}, \mathfrak{h})$, and Δ_+ a choice of the set of positive roots, and $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ the set of simple roots in Δ_+. Take a Chevalley basis $x_\alpha (\alpha \in \Delta)$, $h_1, \cdots, h_l \in \mathfrak{h}$.

Proposition 3.1 [6]. *The normalizer N of H in G is generated by the elements $w_\alpha(s):=\exp(sx_\alpha)\exp(-s^{-1}x_{-\alpha})\exp(sx_\alpha)$ with $\alpha\in\Delta$, and $s\in C^\times$. For any $\alpha\in\Delta$, $s\in C^\times$, the element $h_\alpha(s):=w_\alpha(1)^{-1}w_\alpha(s)$ belongs to H, and the map $(s_1,\cdots,s_l)\to h_{\alpha_1}(s_1)\cdots h_{\alpha_l}(s_l)$ defines a Lie group isomorphism of $(C^\times)^l$ onto H.*

We can define an involutive antilinear antiautomorphism on $\mathfrak{g}$ by $x_{\alpha_i}^*:=x_{\alpha_i}$, $h_i^*:=h_i$ for $i=1,\cdots,l$. We extend this involution and the Killing form $B(\cdot,\cdot)$ on $\mathfrak{g}$ to $\tilde{\mathfrak{g}}_k$, by

$$f^*(s):=f(s)^*,\quad B(f,g):=\int_0^1 B(f(e^{2\pi\sqrt{-1}r}),\ g(e^{2\pi\sqrt{-1}r}))dr \qquad \text{for } f,g\in\tilde{\mathfrak{g}}_k,\ s\in S^1.$$

Then, $f\to f^*$ is an involutive antilinear antiautomorphism, $B(\cdot,\cdot)$ is a non-degenerate invariant symmetric bilinear form.

From now on, we always assume $k\geqq 1$. Put $Z(f,g):=B(\partial f,g)$ for $f,g\in\tilde{\mathfrak{g}}_k$, where ∂ is the derivation on $\tilde{\mathfrak{g}}_k$ defined in the same way as in § 1.

Lemma 3.2. *$Z(\cdot,\cdot)$ is a continuous 2-cocycle on $\tilde{\mathfrak{g}}_k$.*

Let $\hat{\mathfrak{g}}_k$ be the 1-dimensional central extension of $\tilde{\mathfrak{g}}_k$ corresponding to the 2-cocycle $Z(\cdot,\cdot)$, and $[\cdot,\cdot]$ be the bracket product on $\hat{\mathfrak{g}}_k$. As a vector space, $\hat{\mathfrak{g}}_k$ is equal to the direct sum $\tilde{\mathfrak{g}}_k+Cc$, where c represents $1\in C$. Thanks to the continuity of $Z(\cdot,\cdot)$, $\hat{\mathfrak{g}}_k$ becomes naturally a Banach Lie algebra. We remark that the 2-cocycle $Z(\cdot,\cdot)$ coincides with the usual one (cf., [3, 7.1] and [2, 1.1]) on the dense subalgebra $\tilde{\mathfrak{g}}$. Hence, the Banach Lie algebra $\hat{\mathfrak{g}}_k$ contains densely the usual affine Lie algebra $\hat{\mathfrak{g}}$.

§ 4. An action of $\tilde{G}_k$ on $\hat{\mathfrak{g}}_k$. Let k be a positive integer. Denote by $[\cdot,\cdot]_0$ the bracket product on $\tilde{\mathfrak{g}}_k$, and by Ad_0 the adjoint action of $\tilde{G}_k$ on the loop algebra $\tilde{\mathfrak{g}}_k$.

Lemma 4.1. *For each g in $\tilde{G}_k$, there exists a unique element z_g in $\tilde{\mathfrak{g}}_{k-1}$ such that*

$$Z(\mathrm{Ad}_0(g)x,\mathrm{Ad}_0(g)y)=Z(x,y)+B(z_g,[x,y]_0)\qquad \textit{for } \forall x,\ y\in\tilde{\mathfrak{g}}_k.$$

For each $g\in\tilde{G}_k$, define a linear operator $\mathrm{Ad}(g)$ on $\hat{\mathfrak{g}}_k$ by

$$\mathrm{Ad}(g)(x+rc):=\mathrm{Ad}_0(g)x+(B(z_g,x)+r)c\qquad \text{for } x\in\tilde{\mathfrak{g}}_k,\ r\in C.$$

Theorem 4.2. $\mathrm{Ad}:g\to\mathrm{Ad}(g)$ *is a group-homomorphism of $\tilde{G}_k$ into the group* $\mathrm{Aut}(\hat{\mathfrak{g}}_k)$ *of homeomorphic automorphisms on $\hat{\mathfrak{g}}_k$, and is of class C^∞.*

Since the 2-cocycle $Z(\cdot,\cdot)$ is ∂-invariant: $Z(\partial x,y)+Z(x,\partial y)=0$ for $\forall x,\ y\in\tilde{\mathfrak{g}}_k$, ∂ defines, by $\partial c=0$, a continuous linear map from $\hat{\mathfrak{g}}_k$ into $\hat{\mathfrak{g}}_{k-1}$, denoted by the same symbol ∂. It satisfies the derivation property $[\partial x,y]+[x,\partial y]=0$ for $\forall x,\ y\in\hat{\mathfrak{g}}_k$.

We put $\hat{\mathfrak{g}}_k^e:=C\partial+\hat{\mathfrak{g}}_k$, and extend the bracket product on $\hat{\mathfrak{g}}_k$ to a bilinear map $\hat{\mathfrak{g}}_k^e\times\hat{\mathfrak{g}}_k^e\ni(x,y)\to[x,y]\in\hat{\mathfrak{g}}_{k-1}^e$ by

$$[r_1\partial+x_1,r_2\partial+x_2]:=r_1\partial x_2-r_2\partial x_1+[x_1,x_2]\quad \text{for } r_i\in C,\ x_i\in\hat{\mathfrak{g}}_k\ (i=1,2).$$

For each $g\in\tilde{G}_k$, we extend the operator $Ad(g)$ on $\hat{\mathfrak{g}}_k$ to the linear map from $\hat{\mathfrak{g}}_k^e$ into $\hat{\mathfrak{g}}_{k-1}^e$, by

$$\mathrm{Ad}(g)\partial:=\partial+z_{g^{-1}}-\frac{1}{2}B(z_g,z_g)c,$$

Proposition 4.3. *Let* $k \geqq 2$.

i) $[\mathrm{Ad}(g)x, \mathrm{Ad}(g)y] = \mathrm{Ad}(g)[x, y]$ *for* $\forall g \in \tilde{G}_k$, $\forall x, y \in \hat{\mathfrak{g}}_k^e$.

ii) $\mathrm{Ad}(g)\mathrm{Ad}(g')x = \mathrm{Ad}(gg')x$ *for* $\forall g, g' \in \tilde{G}_k$, $\forall x \in \hat{\mathfrak{g}}_k^e$.

§ 5. Weyl group of the completed affine Lie algebra. The dense subalgebra $\hat{\mathfrak{g}}^e := \hat{\mathfrak{g}} + \boldsymbol{C}\partial$ of $\hat{\mathfrak{g}}_k^e$, is one of Kac-Moody algebras of affine type with tier number 1 [3, 7.1], and its Cartan subalgebra is given by $\hat{\mathfrak{h}}^e := \mathfrak{h} + \boldsymbol{C}c + \boldsymbol{C}\partial$. The intersection of $\hat{\mathfrak{h}}^e$ with $\hat{\mathfrak{g}} = [\hat{\mathfrak{g}}^e, \hat{\mathfrak{g}}^e]$ is given by $\hat{\mathfrak{h}} = \mathfrak{h} + \boldsymbol{C}c$. $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^e$ are maximal abelian subalgebras of $\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{g}}_k^e$ respectively. Put $\tilde{H}_k := H(L_k) \subset \tilde{G}_k$. By Proposition 3.1 and Theorem 2.2, we see that

$$(L_k)^\times \times \cdots \times (L_k)^\times \ni (f_1, \cdots, f_l) \longrightarrow h_{\alpha_1}(f_1) \cdots h_{\alpha_l}(f_l) \in \tilde{H}_k$$

is a Lie group isomorphism.

Let $\hat{Z}$ and $\hat{Z}^e$ be centralizers in $\tilde{G}_k$ of $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^e$ respectively. Making use of the results in § 4, we get

Theorem 5.2. $\hat{Z} = \{h_{\alpha_1}(f_1) \cdots h_{\alpha_l}(f_l)\,;\, f_1, \cdots, f_l \in \exp(L_k)\}$, *and* $\hat{Z}^e = H$.

We denote by $\hat{N}$ and $\hat{N}^e$ the normalizers in $\tilde{G}_k$ of $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^e$ respectively.

Theorem 5.3. $\hat{N} = N \cdot \tilde{H}_k$, *and*

$\tilde{N}^e = \hat{N}^e = N \cdot \{h_{\alpha_1}(C_1 \zeta^{n_1}) \cdots h_{\alpha_l}(C_l \zeta^{n_l})\,;\, n_1, \cdots, n_l \in \boldsymbol{Z},\ C_1, \cdots, C_l \in \boldsymbol{C}^\times\}$, *where* $\zeta : S^1 \to S^1 \subset \boldsymbol{C}$ *is the identity map.*

Put $\hat{W} := \hat{N}^e / \hat{Z}^e$ and

$$T := \{h_{\alpha_1}(C_1 \zeta^{n_1}) \cdots h_{\alpha_l}(C_l \zeta^{n_l})\,;\, n_1, \cdots, n_l \in \boldsymbol{Z},\ C_1, \cdots, C_l \in \boldsymbol{C}^\times\}.$$

Since $N \cap T = H = \hat{Z}^e$ and N normalizes T, we have $\hat{W} = (N/H) \ltimes (T/H)$. We see that the mapping $T \ni g \to z_g$ gives an isomorphism of T/H onto the coroot lattice of $(\mathfrak{g}, \mathfrak{h})$. Thus,

Theorem 5.4. *The quotient groups* $\hat{N}/\hat{Z}$ *and* $\hat{N}^e/\hat{Z}^e$ *are both isomorphic to the affine Weyl group* $\hat{W} = W \ltimes \check{Q}$ *canonically, where* W *and* $\check{Q}$ *are the Weyl group and the coroot lattice of* $(\mathfrak{g}, \mathfrak{h})$ *respectively.*

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