6. On the Inequalities of Erdös-Turán and Berry-Esseen. II

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This is continued from [1].

5. The ideas of the proofs of the results given in Sections 3 and 4 are similar. Here we shall prove only Theorem 1. The proof is based on some ideas of Sendov [3] and the author [2]. We begin with a well known lemma of Sendov, which he used in the approximation theory.

Lemma 1 ([3], [4]). *Let f be a periodic function with period 1, and let μ be its modulus of nonmonotonicity (on $\boldsymbol{R}$). Suppose also that $x \in \boldsymbol{R}$ and $\delta \geqq 0$. Then:*

(a) *The inequality $f(t) \leqq f(x)+\mu(2\delta)$ holds either for all $t \in [x, x+\delta]$, or for all $t \in [x-\delta, x]$.*

(b) *The inequality $f(t) \geqq f(x)-\mu(2\delta)$ holds either for all $t \in [x, x+\delta]$, or for all $t \in [x-\delta, x]$.*

In what follows, a periodic function K with period 1 is said to be a *kernel* if it is nonnegative, even and $\int_0^1 K(t)dt=1$.

Lemma 2. *Let f be as in Theorem 1, and let μ be its modulus of nonmonotonicity. Suppose also that K is a kernel, and set*

$$\mathcal{K}(f\,;x)=\int_0^1 f(t)K(t-x)dt \qquad \text{for all } x \in \boldsymbol{R}.$$

Then:

(i) *For every $\delta \in [0, 1/2]$,*

$$\|f\| \leqq \mu(4\delta)+\|\mathcal{K}(f, \cdot)\|+2(2\|f\|-\mu(4\delta)) \int_\delta^{1/2} K(t)dt.$$

(ii) *For every $\delta \geqq 1/2$,*

$$\|f\| \leqq \mu(4\delta)+\|\mathcal{K}(f\,; \cdot)\|.$$

Proof. (i) Let $\delta \in [0, 1/2]$ and $x \in \boldsymbol{R}$. First we shall prove that

$$(1) \qquad |f(x)| \int_{-\delta}^{\delta} K(t)dt \leqq \mu(4\delta) \int_{-\delta}^{\delta} K(t)dt+2\|f\| \int_\delta^{1/2} K(t)dt+\|\mathcal{K}(f\,; \cdot)\|.$$

According to Lemma 1-(a) the inequality

$$(2) \qquad f(t) \leqq f(x)+\mu(4\delta)$$

holds either for all $t \in [x, x+2\delta]$, or for all $t \in [x-2\delta, x]$.

Suppose first that (2) holds for all $t \in [x, x+2\delta]$. In this case we shall obtain an upper bound for the value of $\mathcal{K}(f\,; x+\delta)$. We have

$$(3) \qquad \mathcal{K}(f\,; x+\delta)=\int_{-1/2}^{1/2} f(t+x+\delta)K(t)dt$$

since f is a periodic function with period 1. Now we write $\mathcal{K}(f\,; x+\delta)$ in the form

$$(4)\qquad \mathcal{K}(f\,;x+\delta)=\int_{-\delta}^{\delta} f(t+x+\delta)K(t)dt+\left(\int_{-1/2}^{-\delta}+\int_{\delta}^{1/2}\right)f(t+x+\delta)K(t)dt$$
$$=I_1+I_2,$$

where the meanings of I_1 and I_2 are clear.

Note that if $t\in[-\delta,\delta]$, then

$$x\leqq t+x+\delta\leqq x+2\delta.$$

Hence, from (2) we conclude that

$$(5)\qquad f(t+x+\delta)\leqq f(x)+\mu(4\delta)$$

for these values of t. From the last inequality, we get

$$(6)\qquad I_1\leqq(f(x)+\mu(4\delta))\int_{-\delta}^{\delta} K(t)dt.$$

On the other hand, it is easy to see that

$$(7)\qquad I_2\leqq 2\|f\|\int_{\delta}^{1/2} K(t)dt$$

since $\delta\in[0,1/2]$. Combining (4), (6) and (7), we obtain

$$(8)\qquad \mathcal{K}(f\,;x+\delta)\leqq(f(x)+\mu(4\delta))\int_{-\delta}^{\delta} K(t)dt+2\|f\|\int_{\delta}^{1/2} K(t)dt,$$

which implies

$$(9)\qquad -f(x)\int_{-\delta}^{\delta} K(t)dt\leqq\mu(4\delta)\int_{-\delta}^{\delta} K(t)dt+2\|f\|\int_{\delta}^{1/2} K(t)dt+\|\mathcal{K}(f\,;\cdot)\|.$$

Now suppose that (5) holds for all $t\in[x-2\delta,\,x]$. Then using the same method as in the first alternative we can show the validity of (8) but with $\mathcal{K}(f\,;x-\delta)$ in place of $\mathcal{K}(f\,;x+\delta)$, from which we again arrive at (9).

Further, using Lemma 1-(b) and repeating all the above arguments we can obtain (9) but with $f(x)\int_{-\delta}^{\delta} K(t)dt$ in the left-hand side. Thus the inequality (1) is proved.

Since x is an arbitrary real number, we can replace $f(x)$ in (1) with $\|f\|$. Then the new inequality can be written in the form

$$\left(1-2\int_{\delta}^{1/2} K(t)dt\right)\|f\|\leqq\left(1-2\int_{\delta}^{1/2} K(t)dt\right)\mu(4\delta)+2\|f\|\int_{\delta}^{1/2} K(t)dt+\|\mathcal{K}(f\,;\cdot)\|,$$

which implies the desired inequality in case of $\delta\in[0,1/2]$.

(ii) Now let $\delta\geqq 1/2$ and $x\in\boldsymbol{R}$. To prove the desired inequality it is sufficient to show that

$$(10)\qquad |f(x)|\leqq\mu(4\delta)+\|\mathcal{K}(f\,;\cdot)\|.$$

Let us consider again the inequality (2). Suppose first that it holds for all $t\in[x,x+2\delta]$. Now note that if $t\in[-1/2,\,1/2]$ then $t\in[-\delta,\delta]$, and so (5) holds for $t\in[-1/2,\,1/2]$. From (5) and (3), we deduce

$$\mathcal{K}(f\,;\,x+\delta)\leqq(f(x)+\mu(4\delta))\int_{-1/2}^{1/2} K(t)dt=f(x)+\mu(4\delta),$$

which implies the inequality

$$(11)\qquad -f(x)\leqq\mu(4\delta)+\|\mathcal{K}(f\,;\cdot)\|.$$

If (2) holds for all $t\in[x-\delta,x]$, then we estimate $\mathcal{K}(f\,;x-\delta)$ and again arrive at (11).

Analogously, we can prove (11) with $f(x)$ in place of $-f(x)$, and so (10) is proved. Q.E.D.

In what follows, for an integrable function f on $[0,1]$ and a positive integer m, we denote by $\sigma_m(f)$ the *mth Fejér integral* of f, i.e.,

$$\sigma_m(f\,;x)=\int_0^1 f(t)F_m(t-x)dt \qquad \text{for all } x\in \boldsymbol{R},$$

where

$$F_m(t)=\frac{1}{m}\left(\frac{\sin \pi mt}{\sin \pi t}\right)^2$$

is the mth *Fejér kernel**). We note that for every $\delta\in[0,1/2]$,

$$\int_\delta^{1/2} F_m(t)dt \leqq \frac{1}{m}\int_\delta^{1/2}\frac{dt}{\sin^2\pi t}=(\cot \pi\delta)/(\pi m)<1/(\pi^2 m\delta). \tag{12}$$

Lemma 3. *Let f be as in Theorem* 1, *and let μ be its modulus of non-monotonicity. Then for every positive integer m and every real $a>1$, we have*

$$\|f\|\leqq \frac{a+1}{2}\mu\left(\frac{16a}{\pi^2(a-1)m}\right)+a\|\sigma_m(f\,;\,\cdot)\|. \tag{13}$$

Proof. Let $m\in \boldsymbol{N}$ and $a>1$. We can suppose that

$$\|f\|>\frac{a+1}{2}\mu\left(\frac{16a}{\pi^2(a-1)\mathrm{m}}\right) \tag{14}$$

since otherwise there is nothing to prove. Now set

$$\delta=\frac{4a}{\pi^2(a-1)m}. \tag{15}$$

From (14) and (15), we conclude that

$$2\|f\|-\mu(4\delta)>a\mu(4\delta)\geqq 0. \tag{16}$$

Suppose first that $\delta\in[0,1/2]$. Applying Lemma 2-(i) to the mth Fejér kernel we obtain

$$\|f\|\leqq \mu(4\delta)+\|\sigma_m(f\,;\,\cdot)\|+2(2\|f\|-\mu(4\delta))\int_\delta^{1/2}F_m(t)dt.$$

From this, (12) and (16), we get

$$\|f\|\leqq \mu(4\delta)+\|\sigma_m(f\,;\,\cdot)\|+2\,(2\|f\|-\mu(4\delta))/(\pi^2 m\delta).$$

The last inequality can be written in the form

$$(1-4/(\pi^2 m\delta))\|f\|\leqq (1-2/(\pi^2 m\delta))\,\mu(4\delta)+\|\sigma_m(f\,;\,\cdot)\|,$$

which according to (15) coincides with

$$\frac{1}{a}\|f\|\leqq \frac{a+1}{2a}\mu(4\delta)+\|\sigma_m(f\,;\,\cdot)\|,$$

and so (13) is proved in case of $\delta\in[0,1/2]$.

Now suppose that $\delta\geqq 1/2$. Applying Lemma 2-(ii) to the mth Fejér kernel we get

$$\|f\|\leqq \mu(4\delta)+\|\delta_m(f\,;\,\cdot)\|\leqq \frac{a+1}{2}\mu(4\delta)+a\|\sigma_m(f\,;\,\cdot)\|,$$

which coincides with (13). Q.E.D.

Lemma 4. *Let f be as in Theorem* 1, *and let μ be its modulus of non-monotonicity. Suppose also that $\int_0^1 f(t)dt=0$. Then for every positive*

*) As usual the mth Fejér kernel equals m if t is an integer.

integer m and every real $a>1$, we have

$$\|f\| \leqq \frac{a+1}{2}\mu\left(\frac{16a}{\pi^2(a-1)m}\right)+\frac{a}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h}-\frac{1}{m}\right)|\hat{f}(h)|.$$

Proof. Let $m\in N$ and $a>1$. According to Lemma 3 it is sufficient to show that

$$\|\sigma_m(f\,;\,\cdot)\| \leqq \frac{1}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h}-\frac{1}{m}\right)|\hat{f}(h)|.$$

A proof of the last inequality is given in [2]. Q.E.D.

Proof of Theorem. Let $m\in N$ and $a>1$. It is easy to see that the function φ defined on $\boldsymbol{R}$ by

$$\varphi(x)=f(x)-\int_0^1 f(t)dt$$

satisfies the conditions of Lemma 4, i.e., φ is periodic with period 1, Riemann-integrable on [0, 1], and $\int_0^1 \varphi(t)dt=0$. Therefore, from Lemma 4 we have

$$\|\varphi\| \leqq \frac{a+1}{2}\mu\left(\varphi\,;\frac{16a}{\pi^2(a-1)m}\right)+\frac{a}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h}-\frac{1}{m}\right)|\hat{f}(h)|.$$

Now taking into account that $[f]=[\varphi]\leqq 2\|\varphi\|$, $\mu(f\,;\delta)\equiv\mu(\varphi\,;\delta)$ and $\hat{f}(h)=\hat{\varphi}(h)$, we get the desired inequality for the oscillation of f. Q.E.D.

References

[1] P. D. Proinov: On the inequalities of Erdös-Turán and Berry-Esseen. I. Proc. Japan Acad., **64**A, 381–384 (1988).

[2] ——: On the Erdös-Turán inequality on uniform distribution. I, II. Proc. Japan Acad., **64**A, 27–28, 49–52 (1988).

[3] Bl. Sendov: On some linear methods for approximation of periodic functions in Hausdorff metric. Annuaire Univ. Sofia Fac. Math., **58**, 107–140 (1965) (in Bulgarian).

[4] ——: Some problems of approximation theory of functions and sets in Hausdorff metric. Uspehi Mat. Nauk, **24**, 147–178 (1969) (in Russian).