42. On Escobales-Parker's Theorem

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(Communicated by Kunihiko KODAIRA, M. J. A., May 12, 1989)

Let $\pi: M \to B$ be a Riemannian submersion and the dimension of M be n. The fibre $\pi^{-1}(b)$ for each point b in B is a submanifold of M of dimension $p=n-\dim B$ and denoted by $V(b)$ or generally by V. A vector field on M is vertical or horizontal if it is always tangent or orthogonal to fibres respectively, and the vertical and horizontal components of a vector field E on M are denoted by VE and HE respectively. All compact manifolds are assumed to be without boundary and g is a Riemannian metric on M.

The (1, 2)-tensor fields T and A on M are defined by

$$T_E F = HD_{VE} VF + VD_{VE} HF,$$
$$A_E F = HD_{HE} VF + VD_{HE} HF,$$

for any vector fields E and F on M, where D indicates the covariant derivative on M. Throughout this paper, U, V, W, W' will always be vertical vector fields and X, Y, Z basic ones.

The component $T_U V$ is the second fundamental form of each fibre and $A_X Y$ is related to the obstruction to integrability of horizontal distribution. Each fibre is totally geodesic if $T=0$ and totally umbilic if there exists a horizontal vector field N such that $T_U V = g(U, V)N$ for all U and V. Since $A_X Y = V[X, Y]/2$, the component A is identically zero if and only if the horizontal distribution H is integrable. A. Besse [1, pp. 265-266] proved

Theorem A. *The tensorial invariants of a submersion π satisfying*

(1) $A=0$,

(2) $T_U V = g(U, V)N$,

(3) *N is basic*

characterize locally warped products among Riemannian submersions.

R. H. Escobales and P. E. Parker [3] have recently proved

Theorem B. *Let M be a connected n-dimensional manifold with totally umbilic fibres of dimension $p \leq n-1$. Assume either that M is compact or that all the fibres are compact. Assume further that the normal curvature 1-form N is closed (or, equivalently, under the compactness hypothesis above, assume that the normal curvature vector is basic). Suppose that at each point of M for every vertical vector W,*

(*) $VS(W, W) \leq (1-p)\|N\|^2$, *and at (at least) one point on each fibre,* $VS(W, W) < (1-p)\|N\|^2$,

where $VS(W, W)$ is the partial Ricci curvature in the direction of W. Then the horizontal distribution H is integrable with totally geodesic leaves and locally M is a warped product.

Let R and $\bar{R}$ denote the curvature tensor of M and V. Using the Gauss equation for the Riemannian submersion with totally umbilic fibres, we get

$$g(R(U, V)W, W')=g(\bar{R}(U, V)W, W')+\{g(U, W)g(V, W') - g(V, W)g(U, W')\}\|N\|^2.$$

We donote by $\{V_r\}_{1\le r\le p}$ a local orthonormal basis of vertical vector fields. From the above equation, we obtain

$$g(\bar{R}(V_r, A_X Y)A_X Y, V_r)=g(R(V_r, A_X Y)A_X Y, V_r) +\{\|V_r\|^2\|A_X Y\|^2-\|g(A_X Y, V_r)\|^2\}\|N\|^2.$$

Summing over r, we have

(1) $$\bar{S}(A_X Y, A_X Y)=VS(A_X Y, A_X Y)+(p-1)\|A_X Y\|^2\|N\|^2,$$

where $VS(W, W)=\sum_r g(R(V_r, W)W, V_r)$ and $\bar{S}$ is the Ricci curvature of the fibre V. The Ricci curvature is called quasi-negative if it is negative semidefinite all over a manifold and negative definite at one point of the manifold. By means of (1), we can see that the condition (*) is equivalent for all the fibres to have the quasi-negative Ricci curvature.

The purpose of this note is to prove the following generalization of the Theorem B by a simple method.

Theorem. *Let $\pi: M\to B$ be a Riemannian submersion with totally umbilic fibres V. If all the fibres V are compact and have the quasi-negative Ricci curvature, then M is locally a warped product.*

Proof. Let $\bar{\nabla}$ be the Riemannian connection of V. The following equation is known [1, 2, 3];

$$g((D_V A_X Y), W)+g(V, (D_W A_X Y))=-g(V, W)dN(X, Y),$$

for the Riemannian submersion with totally umbilic fibres. By means of the formula $D_V W=T_V W+\bar{\nabla}_V W$, we see that $g(D_V A_X Y, W)=g(\bar{\nabla}_V A_X Y, W)$, and obtain the equation

(2) $$g((\bar{\nabla}_V A_X Y), W)+g(V, (\bar{\nabla}_W A_X Y))=-g(V, W)dN(X, Y).$$

This means that $A_X Y$ is a conformal Killing vector on each fibre V. K. Yano [4, pp. 38-39] proved that, in a compact space, a conformal Killing vector has a vanishing covariant derivatives if the Ricci curvature is negative semidefinite, and does not exist other than zero vector if the Ricci curvature is negative definite. Therefore the assumption of quasi-negativeness of the Ricci curvature implies that $\bar{\nabla}_V A_X Y=0$ and there is one point q in V at which $(A_X Y)(q)=0$.

If we put

$$f=\|A_X Y\|^2,$$

then f is a non-negative constant on the fibre V and $f(q)=0$ at q in V and hence $f=0$ on the fibre V. Therefore we have $A_X Y=0$ for any basic vectors X and Y, and $A=0$ in M. Moreover N is basic, because N is closed by the equation (2) and any closed horizontal 1-form is basic [3]. Thus the hypotheses of Theorem A are all satisfied, that is, M is locally a warped product.

References

[1] A. Besse: Einstein Manifolds. Springer-Verlag, Berlin (1987).

[2] R. H. Escobales, Jr.: The integrability tensor for bundle like foliations. Trans. Amer. Math. Soc., **270**, 333–339 (1982).

[3] R. H. Escobales, Jr. and P. E. Parker: Geometric consequences of the normal curvature cohomology class in umbilic foliations. Indiana Univ. Math. J., **37**, 389–408 (1988).

[4] K. Yano: Differential Geometry on Complex and Almost Complex Spaces. Pergamon Press, New York (1965).