41. *On the Structure and the Homology of the Torelli Group*

By Shigeyuki MORITA
Department of Mathematics, Tokyo Institute of Technology

(Communicated by Kunihiko KODAIRA, M. J. A., May 12, 1989)

1. Introduction. In a series of papers [2], [3], [4], [5], Johnson developed a detailed study of the structure of the Torelli group and obtained several fundamental results. In our papers [12], [13] we have combined his method with our own [9], [10], [11] and clarified the relationship between the Casson invariant of oriented homology 3-spheres and the structure of certain subgroups of the mapping class groups of orientable surfaces. The purpose of the present note is to announce our recent results along the lines of the above mentioned works. More precisely, on the one hand we show a procedure which enhances Johnson's method of investigating the structure of the mapping class groups and on the other hand we obtain a certain information about the second homology of the Torelli group.

Details will appear elsewhere.

2. A refined from of Johnson's homomorphisms. Here we first recall the definition of Johnson's homomorphisms (see [3], [12] for details) and then we present a refined form of them. Let Σ_g be a closed oriented surface of genus $g \geq 2$ and let $\Gamma_1 = \pi_1(\Sigma_g \setminus \mathrm{Int}\, D^2)$ which is a free group of rank $2g$. Define inductively $\Gamma_{k+1} = [\Gamma_k, \Gamma_1]$ $(k=1, 2, \cdots)$ and put $N_k = \Gamma_1/\Gamma_k$ which we call the k-th nilpotent quotient of Γ_1. We write H for $N_2 = H_1(\Sigma_g; \boldsymbol{Z})$. It is well known that the graded Lie algebra $\oplus \Gamma_k/\Gamma_{k+1}$ is naturally isomorphic to the graded Lie algebra $\mathcal{L} = \oplus \mathcal{L}_k$ freely generated by the elements of H over $\boldsymbol{Z}$ (see [6]). Now let $\mathcal{M}_{g,1}$ be the mapping class group of Σ_g relative to an embedded disc $D^2 \subset \Sigma_g$. $\mathcal{M}_{g,1}$ acts on N_k and let $\mathcal{M}(k)$ be the subgroup of $\mathcal{M}_{g,1}$ consisting of all the elements which act on N_k trivially. $\mathcal{M}(2)$ is nothing but the Torelli group $\mathcal{I}_{g,1}$. Now Johnson's homomorphism

$$\tau_k : \mathcal{M}(k) \longrightarrow \mathrm{Hom}\,(H, \mathcal{L}_k) \simeq H \otimes \mathcal{L}_k$$

is defined as $\tau_k(\varphi)(u) = [\varphi(\gamma)\gamma^{-1}]$ $(\varphi \in \mathcal{M}_{g,1}, u \in H)$, where $\gamma \in \Gamma_1$ is any element such that $[\gamma] = u$ and $[\varphi(\gamma)\gamma^{-1}]$ denotes the image in $\mathcal{L}_k$ of the element $\varphi(\gamma)\gamma^{-1} \in \Gamma_k$.

Now we describe our refined form of the homomorphism τ_k. Choose free generators $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g$ of Γ_1 such that $\pi_1(\Sigma_g)$ is obtained from Γ_1 by adding one relation $\zeta = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = id$. Also choose a 2-chain $\sigma_0 \in C_2(\Gamma_1)$ of the group Γ_1 such that $\partial \sigma_0 = -(\zeta)$ and that its image in $C_2(\pi_1(\Sigma_g))$ is a fundamental cycle. Now let $\varphi \in \mathcal{M}(k)$. Then $\sigma_0 - \varphi_*(\sigma_0)$ is a 2-cycle of the free group Γ_1 so that there exists a 3-chain $c_\varphi \in C_3(\Gamma_1)$ such that $\partial c_\varphi = \sigma_0 - \varphi_*(\sigma_0)$ (we can construct such a 3-chain explicitly by making use of the

Fox free differential calculus). Now by the assumption that φ does not act on N_k, the image $\bar{c}_\varphi$ of c_φ in $C_3(N_k)$ is a cycle. Hence we have the homology class $[\bar{c}_\varphi] \in H_3(N_k)$. Next let $\{E^r_{p,q}\}$ be the spectral sequence of the homology of the central extension $0 \to \mathcal{L}_k \to N_{k+1} \to N_k \to 1$ and let d^2: $E^2_{3,0} \cong H_3(N_k) \to E^2_{1,1} \cong H \otimes \mathcal{L}_k$ be the differential.

Theorem 1. *The correspondence $\tilde{\tau}_k: \mathcal{M}(k) \to H_3(N_k)$ defined by $\tilde{\tau}_k(\varphi) = [\bar{c}_\varphi]$ is a well defined homomorphism. Moreover we have $d^2 \circ \tilde{\tau}_k = \tau_k$.*

Now let $\mathcal{H}_k \subset \mathrm{Hom}(H, \mathcal{L}_k)$ be the kernel of the natural homomorphism $\mathrm{Hom}(H, \mathcal{L}_k) \cong H \otimes \mathcal{L}_k \to \mathcal{L}_{k+1}$ defined by $u \otimes \xi \to [u, \xi]$.

Corollary 2. *The image of Johnson's homomorphism $\tau_k: \mathcal{M}(k) \to \mathrm{Hom}(H, \mathcal{L}_k)$ is contained in $\mathcal{H}_k$.*

3. Graded Lie algebras over Z. First we define a structure of a Lie algebra (over Z) on the graded module $\oplus_{k\geq 2} \mathrm{Hom}(H, \mathcal{L}_k)$ such that the graded submodule $\oplus_{k\geq 2} \mathcal{H}_k$ becomes a Lie subalgebra. To do this we define a bilinear mapping $\mathrm{Hom}(H, \mathcal{L}_k) \otimes \mathcal{L}_l \to \mathcal{L}_{k+l-1}$ by

$$\varphi\{\xi\} = \sum_{i=1}^{l} [\cdots[u_1, u_2], \cdots], \varphi(u_i)], \cdots], u_l]$$

where $\varphi \in \mathrm{Hom}(H, \mathcal{L}_k)$, $\xi = [\cdots[u_1, u_2], \cdots], u_l] \in \mathcal{L}_l$. Note that $\varphi\{\xi\} = \varphi(\xi)$ for $l=1$. Next define a bilinear mapping

$$[\ ,\ ]: \mathrm{Hom}(H, \mathcal{L}_k) \otimes \mathrm{Hom}(H, \mathcal{L}_l) \longrightarrow \mathrm{Hom}(H, \mathcal{L}_{k+l-1})$$

by $[\varphi, \psi](u) = \varphi\{\psi(u)\} - \psi\{\varphi(u)\}$ ($\varphi \in \mathrm{Hom}(H, \mathcal{L}_k)$, $\psi \in \mathrm{Hom}(H, \mathcal{L}_l)$, $u \in H$).

Proposition 3. *The above operation defines a structure of a Lie algebra over Z on the graded module $\oplus_{k\geq 2} \mathrm{Hom}(H, \mathcal{L}_k)$ such that the graded submodule $\oplus_{k\geq 2} \mathcal{H}_k$ is a Lie subalgebra.*

Proposition 4. *The commutator group $[\mathcal{M}(k), \mathcal{M}(l)]$ is contained in $\mathcal{M}(k+l-1)$ so that the graded module $\oplus_{k\geq 2} \mathcal{M}(k)/\mathcal{M}(k+1)$ admits a natural structure of a Lie algebra over Z.*

Theorem 5. *Johnson's homomorphism $\{\tau_k\}_{k\geq 2}$ induce an injective homomorphism $\oplus_{k\geq 2} \mathcal{M}(k)/\mathcal{M}(k+1) \to \oplus_{k\geq 2} \mathcal{H}_k$ of graded Lie algebras.*

It is a very important problem to determine the image of the above homomorphism.

4. Homology of the Torelli group. In [5] Johnson determined the first homology group of the Torelli group and as for higher homologies he raised the following question ([3], p. 174, C). Let $\mathcal{I}_{g,*}$ be the Terelli group of Σ_g relative to the base point. Then there is a natural homomorphism

$$\eta_k: H_k(\mathcal{I}_{g,*}) \longrightarrow H_{k+2}(H) \cong \Lambda^{k+2}(H)$$

which we may call the "Gysin homomorphism", defined as follows. Namely it is the composition $H_k(\mathcal{I}_{g,*}) \to H_{k+2}(\pi_1(\Sigma_g) \rtimes \mathcal{I}_{g,*}) \to H_{k+2}(H)$, where the first map is the usual Gysin homomorphism of the extension $1 \to \pi_1(\Sigma_g) \to \pi_1(\Sigma_g) \rtimes \mathcal{I}_{g,*} \to \mathcal{I}_{g,*} \to 1$, while the second map is induced from the natural homomorphism $\pi_1(\Sigma_g) \rtimes \mathcal{I}_{g,*} \to H$. We have the equality $\eta_1 = \tau_2$ so that $\eta_1 \otimes \boldsymbol{Q}$ is an isomorphism. Johnson asks whether this is the case for all $k \geq 2$ or not. The results of [11] imply that we can also define a similar homomorphism $\bar{\eta}_k: H_k(\mathcal{I}_g) \to \Lambda^{k+2}(H)$, by using the extension $1 \to \pi_1(\Sigma_g) \to \mathcal{I}_{g,*} \to \mathcal{I}_g \to 1$ and the homomorphism $k: \mathcal{I}_{g,*} \to H$ (see [11]).

By making use of a procedure which realizes the Gysin homomorphisms at the levels of cycles in the context of group homology and also of the free differential calculus, we can determine the values of the above homomorphisms at any given cycle of the Torelli group. In particular we have the following result part of which answers the question of Johnson mentioned above negatively for $k=2$.

Theorem 6. (i) *The Gysin homomorphism* $\eta_2 \otimes \boldsymbol{Q}: H_2(\mathcal{I}_{g,*}; \boldsymbol{Q}) \to \Lambda^4(H) \otimes \boldsymbol{Q}$ *is a surjection for any* $g \geq 2$, *but it is far from being injective for all* $g \geq 3$.

(ii) $\bar{\eta}_1(\varphi) = p^2\{p\tau_2(\tilde{\varphi}) + \theta \wedge C(\tau_2(\tilde{\varphi}))\}$, *where* $\tilde{\varphi} \in \mathcal{I}_{g,*}$ *is any lift of* $\varphi \in \mathcal{I}_g$, $p = 2-2g$, $\theta \in \Lambda^2(H)$ *is the "symplectic class" and* $C: \Lambda^3 H \to H$ *is the contraction* (*see* [2] *for the definition of* θ *and* C), *so that* $\bar{\eta}_1$ *is an injection modulo torsions.*

At present we do not know whether the Gysin homomorphisms $\eta_k \otimes \boldsymbol{Q}$ are surjective for $k \geq 3$ or not. This problem is closely related with the non-triviality of the characteristic classes of the *Torelli space* $\boldsymbol{T}_g$, which is a complex manifold defined to be the quotient of the Teichmüller space $\mathcal{T}_g$ divided by the Torelli group $\mathcal{I}_g$. More precisely, by making use of the Atiyah-Singer index theorem [1] one can show that the "naive" Chern classes of the moduli space $\boldsymbol{M}_g$ of compact Riemann surfaces of genus g can be expressed as a polynomial of the characteristic classes of surface bundles $e_i \in H^{2i}(\boldsymbol{M}_g; \boldsymbol{Q})$ introduced in [7], [9], [14]. If we pull back these classes to the Torelli space, we can prove

Proposition 7. *Let* $s_k(c) \in H^{2k}(\boldsymbol{T}_g; \boldsymbol{Q}) \cong H^{2k}(\mathcal{I}_g; \boldsymbol{Q})$ *be the Chern class of the complex manifold* $\boldsymbol{T}_g$ *corresponding to the formal sum* $\sum t_j^k$ (see [8]). *Then* $s_{2k-1}(c) = 0$ *for all* k *and* $s_{2k}(c)$ *is a multiple of* $e_{2k} \in H^{4k}(\mathcal{I}_g; \boldsymbol{Q})$.

Hence as to the Torelli space, the Pontrjagin classes contain all the informations of the rational Chern classes and it might be worthwhile to know whether these classes are trivial or not. In view of the results of [11], if we know the image of the homomorphism $\bar{\eta}_{2k}$, then we can answer this question.

References

[1] M. F. Atiyah and I. M. Singer: The index of elliptic operators. IV. Ann. Math., **92**, 119–138 (1970).

[2] D. Johnson: An abelian quotient of the mapping class group $\mathcal{I}_g$. Math. Ann., **249**, 225–242 (1980).

[3] ——: A survey of the Torelli group. Contemporary Math., **20**, 165–179 (1983).

[4] ——: The structure of the Torelli group. I. Ann. Math., **118**, 423–442 (1983).

[5] ——: The structure of the Torelli group. II; III. Topology, **24**, 113–144 (1985).

[6] W. Magnus, A. Karrass, and D. Solitar: Combinatorial Group Theory. Interscience Publ. Wiley & Sons (1966).

[7] E. Y. Miller: The homology of the mapping class group. J. Differential Geometry, **24**, 1–14 (1986).

[8] J. Milnor and J. Stasheff: Characteristic Classes. Ann. Math. Studies, no. 76, Princeton Univ. Press (1974).

[9] S. Morita: Characteristic classes of surface bundles. Invent. Math., **90**, 551–577 (1987).

[10] ——: Families of Jacobian manifolds and characteristic classes of surface bundles. I (to appear in Ann. Inst. Fourier).

[11] ——: Families of Jacobian manifolds and characteristic classes of surface bundles. II. Math. Proc. Camb. Phil. Soc., **105**, 79–101 (1989).

[12] ——: Casson's invariant for homology 3-spheres and characteristic classes of surface bundles. I (to appear in Topology).

[13] ——: On the structure of the Torelli group and the Casson invariant (in preparation).

[14] D. Mumford: Towards an enumerative geometry of the moduli space of curves. Arithmetic and Geometry. Progress Math., **36**, 271–328 (1983).