34. Unique Solvability of Nonlinear Fuchsian Equations

By Nour Said MADI*) and Masafumi YOSHINO**)

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1. Introduction. Let $p\geq 2$ and $q\geq 0$ be integers, and let $x=(x_1, \cdots, x_p)$ and $y=(y_1, \cdots, y_q)$ be the variables in C^p and C^q, respectively. We denote by Z and N the set of integers and that of nonnegative integers, respectively. For a multi-index $\alpha=(\alpha_1, \cdots, \alpha_p)\in Z^p$, we set $x^\alpha=x_1^{\alpha_1}\cdots x_p^{\alpha_p}$, $|\alpha|=\alpha_1+\cdots+\alpha_p$.

Let $m\geq 1$. Then we shall prove the unique solvability of nonlinear Fuchsian equations

$$a(x, y\,;\, D_x^\alpha D_y^\beta x^\gamma u\,;\, |\alpha|=|\gamma|\leq m, |\alpha|+|\beta|\leq m)=0, \tag{1}$$

where $a(x, y\,;\, z_{\alpha\beta\gamma})$ is a holomorphic function of x, y and $z=(z_{\alpha\beta\gamma})$. Because the study of the case $p=1$ is classical (cf. [1]), we are interested in the case $p\geq 2$. Madi [3] solved (1) under a so-called Poincare condition if $\alpha=\gamma$ and if (1) is linear. But, in the general case $\alpha\neq\gamma$, the definition of a Poincare condition is not clear. We also have a problem of a derivative loss which is caused by nonlinear terms in (1) such that $\beta\neq 0$.

We shall define a Poincare condition for (1) so that it extends the one in [3] in a natural way. Then we show the existence and uniqueness of solutions of (1) with an additional weak spectral condition (A. 3). A deeper connection between the generalized Poincare condition and the Hilbert factorization problem is also discussed.

The proof is done by a reduction to a system of equations on a scale of Banach spaces, which enables us to estimate the derivative loss of nonlinear terms.

2. Statement of results. We denote by $C_y\{\{x\}\}$ the set of all formal power series $\sum_{\alpha\in N^p} u_\alpha(y)x^\alpha$ where $u_\alpha(y)$ are analytic functions of y in some neighborhood of the origin independent of α. We denote by $C_y\{x\}$ the set of analytic functions of x and y at the origin. For a positive number $a\leq 1$, we define a ball B_a by $B_a=\{y\in C^q\,;\, |y_i|<a, i=1, \cdots, q\}$.

Let $A\subset\{\alpha\in Z^p\,;\, |\alpha|\geq 0\}$ and $B\subset N^q$ be finite sets. Let π be the projection onto $C_y\{\{x\}\}$;

$$\pi x^\alpha u(x, y)=\sum_{\eta, \eta+\alpha\geq 0} u_\eta(y)x^{\eta+\alpha}, \quad u(x, y)=\sum_{\eta\geq 0} u_\eta(y)x^\eta\in C_y\{\{x\}\}. \tag{2}$$

We denote by $p_{\alpha\beta}(\partial)$ $(\alpha\in A, \beta\in B)$ multipliers of order $m_{\alpha\beta}$ given by

$$p_{\alpha\beta}(\partial)v(x, y)=\sum_{\eta\geq 0} v_\eta(y)p_{\alpha\beta}(\eta)x^\eta, \quad v(x, y)=\sum_{\eta\geq 0} v_\eta(y)x^\eta\in C_y\{\{x\}\}, \tag{3}$$

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*) Département de mathématiques, Fac. des Sciences, Rabat, Maroc.
**) Department of Mathematics, Tokyo Metropolitan University, Japan.

where $p_{\alpha\beta}(\eta)$ are symbols of constant-coefficients classical pseudo-differential operators of order $m_{\alpha\beta}$. We denote by $p^0_{\alpha\beta}(\eta)$ the principal part of $p_{\alpha\beta}(\eta)$.

Let $F(x, y, z_{\alpha\beta})$, $\alpha\in A$, $\beta\in B$ be holomorphic in some neighborhood of the origin $x=0$, $y=0$, $z_{\alpha\beta}=0$ such that $F(0, y, 0)\equiv 0$. Then we shall study the solvability and uniqueness of the following equation

(4) $$F(x, y, p_{\alpha\beta}(\partial)\pi x^{\alpha}D_y^{\beta}u\,;\, m_{\alpha\beta}+|\beta|\leq 0)=0.$$

We assume

(A.1) $$p_{\alpha\beta}=0 \quad \text{if } |\alpha|=0,\ \alpha\in A \text{ and } \beta\neq 0.$$

We set $\Phi_{\alpha}(y)=(\partial F/\partial z_{\alpha 0})(0, y, 0)$ if $\alpha\in A$, $=0$ if otherwise. For $\nu=0, 1, 2, \cdots$, let us define the set J_{ν} by

(5) $$J_{\nu}=\{\eta\in \boldsymbol{Z}^p\,;\, \eta=\zeta-(\nu, 0, \cdots, 0),\ \zeta\in \boldsymbol{N}^p,\ |\zeta|=\nu\}.$$

Let n_{ν} be the number of elements of J_{ν}. Then we line up all elements of J_{ν} in some way, and we define the n_{ν} square matrix $\mathcal{B}_{\nu}(y, \eta)$ by

(6) $$\mathcal{B}_{\nu}(y, \eta)=(\Phi_{\mu-\gamma}(y)p^0_{\mu-\gamma,0}(\eta))_{\mu,\gamma\in J_{\nu}}$$

that is, the (μ, γ) component of $\mathcal{B}_{\nu}(y, \eta)$ is given by $\Phi_{\mu-\gamma}(y)p^0_{\mu-\gamma,0}(\eta)$.

Let $|\cdot|_1$ denote the usual l_1-norm on a finite dimensional vector space, that is, the sum of absolute values of all components. We denote by $\bar{\boldsymbol{B}}_a$ the closure of the set $\boldsymbol{B}_a$. Then we assume

(A.2) There exist $c>0$ and $\nu_0\geq 0$ such that, for any $\nu\geq\nu_0$ and $\eta\in\boldsymbol{R}^p$, $|\eta|=1$, and all $y\in\bar{\boldsymbol{B}}_a$, the following estimate holds:

(7) $$|\mathcal{B}_{\nu}(y, \eta)X|_1\geq c|X|_1 \quad \text{for all } X\in \boldsymbol{C}^{n_{\nu}}.$$

(A.3) $$\sum_{\substack{|\alpha|=0,\alpha\neq 0\\ \alpha\in A}} |p_{\alpha,0}(\eta)||\Phi_{\alpha}(0)|<|p_{0,0}(\eta)||\Phi_0(0)| \quad \text{for all } \eta\in \boldsymbol{N}^p.$$

Then our main result is

Theorem 1. *Suppose* (A.1), (A.2), *and* (A.3). *Then the equation* (4) *has a unique solution* $u(x, y)$ *such that* $u(0, y)\equiv 0$ *which is holomorphic in some neighborhood of the origin of* $x=0$, $y=0$.

Example 2 (Fuchsian equations). Let us consider (1). We easily see that, for every α, $\gamma\in \boldsymbol{N}^p$, $|\alpha|=|\gamma|$

$$D_x^{\alpha}x^{\gamma}=\left\{\prod_{j=1}^{p}(\partial_j+\alpha_j)\cdots(\partial_j+1)\right\}\pi x^{\gamma-\alpha}, \quad \partial_j=x_jD_{x_j}, \quad j=1, \cdots, p.$$

We set $v=\Lambda^{-m}u$ where Λ denotes the multiplier with the symbol $(1+|\eta|^2)^{1/2}$. Substituting these relations into (1) we can reduce (1) to (4).

Especially, if $\alpha=\gamma$ in (1), then we can reduce (1) to (4) such that $p_{\alpha\beta}=0$ if $\alpha\neq 0$. Hence, the matrix $\mathcal{B}_{\nu}(y, \eta)$ is a diagonal one, which implies that the conditions (A.2) and (A.3) are equivalent to a so-called Poincare condition. Moreover (A.3) is equivalent to that $p_{0\beta}=0$ if $\beta\neq 0$. Thus, by Theorem 1-(1) has a unique solution if it satisfies a Poincare condition (cf. [3]).

Finally, we assume $p=2$, and we shall give the relation between (A.2) and the factorization of a certain polynomial.

Let $\eta\in\boldsymbol{R}^2$, $|\eta|=1$, $y\in\bar{\boldsymbol{B}}_a$, and let $\mathcal{UA}_{\eta}$ denote the set of functions $g(t, y, \eta)$ of $t\in\boldsymbol{C}$, $|t|=1$ having an absolutely convergent Fourier expansion of $t=e^{i\theta}$ which converges uniformly with respect to $\eta\in\boldsymbol{R}^2$, $|\eta|=1$ and

$y \in \bar{\boldsymbol{B}}_a$. Moreover let $\mathcal{UA}_\eta^+$ (resp. $\mathcal{UA}_\eta^-$) denote the subset of functions of $\mathcal{UA}_\eta$ whose negative (resp. positive) Fourier coefficients vanish. Every function $g_+(t, y, \eta) \in \mathcal{UA}_\eta^+$ (resp. $g_-(t, y, \eta) \in \mathcal{UA}_\eta^-$) can be extended to an analytic function in $|t| \leq 1$ (resp. $|t| \geq 1$) in terms of its Fourier expansion. We denote them with the same letter.

We say that $g(t, y, \eta) \in \mathcal{UA}_\eta$ is uniformly factorizable with respect to $\eta \in \boldsymbol{R}^2$, $|\eta|=1$ and $y \in \bar{\boldsymbol{B}}_a$ if there exist functions $g_\pm(t, y, \eta) \in \mathcal{UA}_\eta^\pm$ such that $g_\pm(t, y, \eta) \neq 0$ for all $(t, y, \eta) \in \{\boldsymbol{C} \times \bar{\boldsymbol{B}}_a \times \boldsymbol{R}^2; |\eta|=1, |t|^{\pm 1} \leq 1\}$, $g_\pm(t, y, \eta)^{-1} \in \mathcal{UA}_\eta^\pm$ and that $g(t, y, \eta) = g_+(t, y, \eta) g_-(t, y, \eta)$.

We set, for $\eta \in \boldsymbol{R}^2$, $|\eta|=1$ and $y \in \bar{\boldsymbol{B}}_a$

$$\sigma_P(t, y, \eta) = \sum_{\alpha, \alpha_1+\alpha_2=0} \Phi_\alpha(y) p_{\alpha_0}^0(\eta) t^{\alpha_1}. \tag{8}$$

Then $\sigma_P(t, y, \eta) \in \mathcal{UA}_\eta$ by definition. Let us define the winding number $I_\eta(\sigma_P)$ of $\sigma_P(t, y, \eta)$ at the origin when t moves on the unit circle S^1 by

$$I_\eta(\sigma_P) = (2\pi)^{-1} \Delta_{t \in S^1} \arg \sigma_P(t, y, \eta). \tag{9}$$

Then we have

Proposition 3. (a) *Suppose $p=2$ and that $\sigma_P(t, y, \eta)$ is uniformly factorizable. Then we have* (A. 2).

(b) *$\sigma_P(t, y, \eta)$ is uniformly factorizable if and only if the following conditions are satisfied*: (i) *$\sigma_P(e^{i\theta}, y, \eta) \neq 0$ for all $0 \leq \theta \leq \pi$, $\eta \in \boldsymbol{R}^2$, $|\eta|=1$, $y \in \bar{\boldsymbol{B}}_a$.* (ii) *$I_\eta(\sigma_P)=0$ for all $\eta \in \boldsymbol{R}^2$, $|\eta|=1$, $y \in \bar{\boldsymbol{B}}_a$.*

References

[1] M. S. Baouendi and C. Goulouic: Cauchy problems with characteristic initial hypersurface. Comm. Pure Appl. Math., **26**, 455–475 (1973).

[2] N. S. Madi: Problème de Goursat holomorphe à variables Fuchsiennes. Bull. Sc. Math., **111**, 291–312 (1987).

[3] ——: Solutions locales pour des opérateurs holomorphes en plusieurs variables (à paraitre).

[4] F. Treves: Ovchyannikov theorem and hyperdifferential operators. Notas de Matematica, **46**, I.M.P.A., Brazil (1968).

[5] C. Wagschal: Une généralisation du problème de Goursat pour des système d'équations intégro-differentielles ou partiellement holomorphes. J. Math. Pures Appl., **53**, 99–132 (1974).

[6] M. Yoshino: Convergence of formal solutions for Fuchs-Goursat equations. J. Diff. Eqs., **74**, 266–284 (1988).