33. *Note on the Reproducing Property of the Bergman Kernel*

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§ 1. In this short note we give a remark that the Bergman kernel of a bounded pseudoconvex domain $\Omega \subset \boldsymbol{C}^n$ with real analytic boundary satisfying Condition Q, has the reproducing property not only on the subspace $H(\Omega)$ of L^2-class holomorphic functions on Ω but also on the whole space $\mathcal{O}(\Omega)$ of functions holomorphic in Ω. The problem only lies on how to formulate this assertion in precise mathematical words. The notion of *mild hyperfunctions* introduced by Kataoka [4] is adequate for this purpose.

Let $B(z, \overline{w})$ denote the Bergman kernel of the domain Ω. According to Bell [1] we shall say that Ω satisfies Condition Q if for any $V \Subset \Omega$ there exists a neighborhood U of $\overline{\Omega}$ such that $B(z, \overline{w})$ is holomorphic in $z, \overline{w}$ in $U \times V^c$, where the superscript c denotes the complex conjugate. (Bell's definition is apparently weaker, but in fact equivalent to this. See lemma in § 2. We wish to pose no such condition, but this property seems still unknown in general. It is well known if Ω is further strictly pseudoconvex.) Let $\mathcal{B}[\overline{\Omega}]$ be the space of hyperfunctions with supports in $\overline{\Omega}$ (which is isomorphic to the dual of the space $\mathcal{A}(\overline{\Omega})$ of real analytic functions on $\overline{\Omega}$). Then we have the following well-defined continuous mapping:

$$
(1) \qquad
\begin{array}{ccc}
\mathcal{B}[\overline{\Omega}] & \longrightarrow & \mathcal{O}(\Omega) \\
\cup\!\!\!| & & \cup\!\!\!| \\
v(z) & \longmapsto & \displaystyle\int_{C^n} B(z, \overline{w}) v(w) |dw|^2,
\end{array}
$$

where $|dw|^2$ denotes the Lebesgue measure of $\boldsymbol{C}^n$. (Similar fact is already remarked by Zorn [7]; He treats complex analytic functionals instead of real analytic ones.) Next, remark that a function u holomorphic in Ω is a solution of the Cauchy-Riemann system which is elliptic everywhere. (To make the argument more elementary, we can consider u to be a solution of the single elliptic equation $\Delta u = 0$ on $\boldsymbol{R}^{2n}$ as is done below.) Hence it is mild at the boundary in the sense of Kataoka [4], and there exists a well defined mapping associating to u its canonical extension:

$$
(2) \qquad
\begin{array}{ccc}
\mathcal{O}(\Omega) & \longmapsto & \mathcal{B}(\overline{\Omega}) \\
\cup\!\!\!| & & \cup\!\!\!| \\
u(z) & \longrightarrow & [u].
\end{array}
$$

(Shortly speaking, the canonical extension is the generalization of making the product $Y(-f(z, \bar{z}))u(z)$, where $Y(t)$ denotes the Heaviside function and f is a real analytic function defining Ω : $\Omega = \{z \in \boldsymbol{C}^n ; f(z, \bar{z}) < 0\}$, with $df \neq 0$

on $\partial\Omega$. This multiplication is obviously meaningful when u is microanalytic on the conormal of $\partial\Omega$. The theory of mild hyperfunctions generalizes this procedure to any hyperfunction solution at a non-characteristic boundary.) The mapping (2) is also continuous in view of the closed graph theorem, as is shown in § 2. Thus the composed mapping $\mathcal{O}(\Omega)\to\mathcal{O}(\Omega)$ of (1) and (2) is continuous. It is equal to the identity on the subspace $H(\Omega)$ by the definition of the Bergman kernel. As is shown in § 2, $H(\Omega)$ is dense in $\mathcal{O}(\Omega)$. Thus we have obtained

Theorem. *Let Ω be a bounded pseudoconvex domain with real analytic boundary satisfying Condition* Q. *Then for any $u\in\mathcal{O}(\Omega)$ and $z\in\Omega$, we have*

$$u(z)=\int_{C^n} B(z,\overline{w})[u(w)]\,|dw|^2,$$

where $[u(w)]$ is the element of $\mathcal{B}[\overline{\Omega}]$ obtained as the canonical extension of $u(z)$.

The composition of the mappings (1), (2) in the reverse order gives a continuous mapping

$$\begin{array}{ccc} \mathcal{B}[\overline{\Omega}] & \longrightarrow & \mathcal{B}[\overline{\Omega}] \\ \cup\!\!| & & \cup\!\!| \\ v & \longmapsto & \left[\int_{C^n} B(z,\overline{w})v(w)\,|dw|^2\right] \end{array} \tag{3}$$

which is an identity on the subspace $[\mathcal{O}(\Omega)]$ of the canonical extensions of elements of $\mathcal{O}(\Omega)$. Thus we obtain

Corollary. *$[\mathcal{O}(\Omega)]$ is a closed subspace of $\mathcal{B}[\overline{\Omega}]$ admitting a topological linear complement. The mapping* (3) *gives the canonical projection $\mathcal{B}[\overline{\Omega}]\to[\mathcal{O}(\Omega)]$.*

Note that this splitting property is not a trivial assertion, because in the general criterion of Vogt [6], $\mathcal{B}[\overline{\Omega}]$ is not a space of type (DN).

§ 2. Now we shall give proofs to the two points assumed above. The first is the continuity of the mapping (2). Assume that $u_k(z)\to u(z)$ in $\mathcal{O}(\Omega)$ and $[u_k]\to v$ in $\mathcal{B}[\overline{\Omega}]$. By the definition of the canonical extension, we have $f(z,\bar{z})^2\Delta[u_k(z)]=0$, where $\Delta=4\sum_{j=1}^n\partial_j\bar{\partial}_j$. Since the operators $f(z,\bar{z})\cdot$, Δ act continuously on $\mathcal{B}(\overline{\Omega})$, passing to the limit we obtain $f(z,\bar{z})^2\Delta v=0$. Thus v is also the canonical extension of a harmonic function $u_\infty(z)$ in Ω. The convergence of $\Delta[u_k]$ to Δv implies that the boundary values of u_k at $\partial\Omega$ converge to the corresponding boundary values of u_∞. Recall here the representation formula for the harmonic function

$$u_k(z)=\int_{\partial\Omega}\left(u_k(z)\Big|_{\partial\Omega}\cdot\frac{\partial E}{\partial\nu}(z)-E(z)\cdot\frac{\partial u_k}{\partial\nu}(z)\Big|_{\partial\Omega}\right)ds,$$

where E denotes the fundamental solution of Δ on $\boldsymbol{R}^{2n}$ and $\partial/\partial\nu$ denotes the exterior normal derivative of $\partial\Omega$. From this formula, in view of the convergence of the boundary values we can see that u_k converges to u_∞ at least pointwise in Ω. Thus $u_\infty(z)=u(z)$, and the mapping (2) has a closed graph. Hence it is continuous.

Recall that $\mathscr{B}[\partial\Omega]$ is dense in $\mathscr{B}[\bar{\Omega}]$, which explains the necessity of the above detailed proof. Here the difficulty of non-locality of the topology is excluded by the fact that the order of the generalized function $\Delta[u_k]$ with respect to the normal variable is bounded by 1.

The second is the denseness of $H(\Omega)$ in $\mathcal{O}(\Omega)$. We shall show this by proving that the subset $\mathcal{O}(\bar{\Omega})\subset H(\Omega)$ is dense in $\mathcal{O}(\Omega)$. We imagine that we will have a simpler proof valid for much wider class of Ω. (For example, the assertion is trivially true irrespectively of the pseudo-convexity if Ω is star-shaped.) But we employ here a deep result of Diederich-Fornaess [2] asserting that for any bounded pseudoconvex domain Ω with real analytic boundary, $\bar{\Omega}$ admits a fundamental system of Stein neighborhoods, say $\{\Omega_l\}_{l=1}^{\infty}$. In view of the Weil-Oka approximation theorem (see e.g. [3], Chap. VIII), it suffices to show that for any compact subset $K\subset\Omega$ its $\mathcal{O}(\Omega_l)$-hull $\hat{K}_{\Omega_l}$ is contained in Ω for some l. We have

$$\operatorname{dis}(\hat{K}_{\Omega_l}, \partial\Omega_l)=\operatorname{dis}(K, \partial\Omega_l)\geqq \operatorname{dis}(K, \partial\Omega),$$

and in view of the regularity of $\partial\Omega$ we have obviously

$$\operatorname{dis}(\hat{K}_{\Omega_l}, \partial\Omega_l)-\operatorname{dis}(\hat{K}_{\Omega_l}, \partial\Omega)\longrightarrow 0 \qquad \text{as } l\longrightarrow\infty.$$

Thus we should have $\hat{K}_{\Omega_l}\subset\Omega$ for $l\gg 1$.

Remark that the denseness of $\mathcal{O}(\bar{\Omega})$ in $\mathcal{O}(\Omega)$ follows in turn from our theorem. In fact, by a result of [5] we have $Y(-\varepsilon-f(z,\bar{z}))u(z)\to[u(z)]$ in $\mathscr{B}[\bar{\Omega}]$, hence $\int_{f(w,\bar{w})\leqq-\varepsilon} B(z,\bar{w})|dw|^2\to u(z)$ in $\mathcal{O}(\Omega)$.

The following lemma asserts the equivalence of Bell's Condition Q with ours.

Lemma. *Assume that for any $\varphi\in C_0^\infty(\Omega)$, $\int_\Omega B(z,\bar{w})\varphi(w)|dw|^2$ extends holomorphically to a neighborhood of $\bar{\Omega}$. Then for any $V\Subset\Omega$ we can find a neighborhood $U\supset\bar{\Omega}$ such that $B(z,\bar{w})$ is holomorphic in $z,\bar{w}$ in $U\times V^c$.*

In fact, let $g(\bar{w})\in\mathcal{O}'(\Omega^c)$. Since Ω^c is Stein, by the Serre duality we have the exact sequence

$$0\longleftarrow\mathcal{O}'(\Omega^c)\longleftarrow C_0^\infty(\Omega^c)\xleftarrow{\ {}^t\bar{\partial}\ }(C_0^\infty(\Omega^c))^n.$$

Hence there exists $\varphi(w)\in C_0^\infty(\Omega)$ such that

$$\langle B(z,\bar{w}), g(\bar{w})\rangle_{\bar{w}}=\int_\Omega B(z,\bar{w})\varphi(w)|dw|^2.$$

Thus we obtain a well-defined mapping

$$\begin{array}{cccc} \beta: & \mathcal{O}'(\Omega^c) & \longrightarrow & \mathcal{O}(\bar{\Omega}) \\ & \in & & \in \\ & g(\bar{w}) & \longmapsto & \int_\Omega B(z,\bar{w})|dw|^2 \end{array} \tag{4}$$

which constitutes the horizontal arrow of the diagram

$$\begin{array}{ccc} \mathcal{O}'(\Omega^c) & \longrightarrow & \mathcal{O}(\bar{\Omega}) \\ & \searrow & \uparrow\cup \\ & & \mathcal{O}(\Omega). \end{array}$$

Since the other mappings are continuous, in view of the closed graph theorem and the unique continuation property we conclude that the mapping

(4) is continuous. Thus a bounded set M of $\mathcal{O}'(\Omega^c)$ is mapped to a bounded set $\beta(M)$ of $\mathcal{O}(\bar{\Omega})$. Especially, for $M=\{\delta(\bar{w}-\bar{a})\,;\, a\in V\}$ with $V\Subset\Omega$, $\beta(M)=\{B(z,\bar{a})\,;\, a\in V\}$ is bounded in $\mathcal{O}(\bar{\Omega})$. Since $\mathcal{O}(\bar{\Omega})=\varinjlim_{U\supset\bar{\Omega}}\mathcal{O}(U)$ is a topological vector space of type (DFS), $\beta(M)\subset\mathcal{O}(U)$ for some U. Namely, the function $B(z,\bar{a})$ of z extends holomorphically to a fixed domain U independent of a. Thus by the Hartogs lemma we can conclude that $B(z,\bar{w})$ is jointly holomorphic in $z,\bar{w}$ on $U\times V^c$.

References

[1] Bell, S. R.: Analytic hypoellipticity of the $\bar{\partial}$-Neumann problem and extendability of holomorphic mappings. Acta Math., **147**, 109–116 (1981).

[2] Diederich, K. and Fornaess, J. E.: Pseudoconvex domains with real analytic boundary. Ann. of math., **107**, 385–397 (1978).

[3] Gunning, R. C. and Rossi, H.: Analytic Functions of Several Complex Variables. Prentice-Hall (1965).

[4] Kataoka, K.: Micro-local theory of boundary value problems. I. J. Fac. Sci. Univ. Tokyo, Sec. IA, **27**, 355–399 (1980).

[5] ——: Some properties of mild hyperfunctions and an application to propagation of micro-analyticity in boundary value problems. ibid., **30**, 279–297 (1983).

[6] Vogt, D.: On the functors $Ext^1(E,F)$ for Fréchet spaces. Studia Math., **85**, 163–197 (1987).

[7] Zorn, P.: Analytic functionals and Bergman spaces. Ann. Scuola Norm. Sup. Pisa, **9**, 365–402 (1982).