31. Yang-Mills-Higgs Fields and Harmonicity of Limit Maps

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Consider a connection A and a Higgs field Φ on the trivial $SU(2)$ bundle over $\boldsymbol{R}^3$, the Euclidean 3-space. A configuration (A, Φ) is called a Yang-Mills-Higgs field if it is a critical point of the action integral $\mathcal{Y}(A, \Phi)=\int_{\boldsymbol{R}^3}\{|F_A|^2+|\nabla_A\Phi|^2\}d^3x$ ($F_A=dA+[A, A]$ and $\nabla_A\Phi+d\Phi+[A, \Phi]$ denote the curvature of A and the covariant derivative of Φ, respectively).

Yang-Mills-Higgs field satisfies the Euler-Lagrange equations $d_A*F+[\Phi, *\nabla_A\Phi]=0$, $d_A(*\nabla_A\Phi)=0$.

The infinity condition on Higgs fields Φ: $|\Phi|(x)\to 1(|x|\to\infty)$ should be posed in order to avoid the trivial case. Then, for each (A, Φ) the degree of the normalized Higgs field at the infinity 2-sphere $\Phi/|\Phi|: S^2_\infty\to S^2\subset\mathfrak{su}(2)$ defines $k\in\boldsymbol{Z}$, called the charge.

A configuration (A, Φ) with finite $\mathcal{Y}(A, \Phi)$ satisfying Bogomolnyi equations, $\nabla_A\Phi=\pm *F_A$, yields a Yang-Mills-Higgs field. We call such a Yang-Mills-Higgs field a magnetic monopole.

Yang-Mills-Higgs fields correspond to 4-dimensional Yang-Mills connections and magnetic monopoles to (anti-)instantons.

Like the moduli space of instantons, the moduli space of charge k monopoles is variously considered. It turns out that the moduli space M_k is a complete hyperkähler manifold ([2]). The twistor formalism was applied by Hitchin and monopoles were transferred into holomorphic structures on a certain complex vector bundle over the space $G(\boldsymbol{R}^3)$ of all oriented lines in $\boldsymbol{R}^3$ and it was further shown that monopoles are interpreted as solutions to Nahm's equations ([5], [6]). By using these, Donaldson proved that M_k is in a one-to-one correspondence to a complex manifold $\mathcal{R}_k$ of all holomorphic maps $f: \boldsymbol{CP}^1\to\boldsymbol{CP}^1$, $f(\infty)=0$, of degree k([3]).

This observation is considered as presentation of a correspondence between the two different variational objects: Yang-Mills-Higgs fields and harmonic maps, because every holomorphic map is harmonic. A harmonic map $f: S^2\to X$ is critical for the energy functional $\mathcal{E}(f)=\int_{S^2}|df|^2 d\sigma$ ([4]).

In this paper we obtain the following phenomenon which gives a more direct representation of Yang-Mills-Higgs fields into harmonic maps by using the limis of Higgs fields at infinity.

Theorem 1. *Let* $[(A, \Phi)]$ *be a gauge equivalence class of* $SU(2)$ *Yang-Mills-Higgs field of* $\mathcal{Y}(A, \Phi)<\infty$ *with the asymptotical condition*

$\lim_{R\to\infty}\sup_{|x|\le R}||\Phi|(x)-1|=0$. *If the representative* (A,Φ) *of* $[(A,\Phi)]$ *in radial gauge satisfies* $\lim_{R\to\infty}\langle R^2A(Rx),\Phi(Rx)\rangle=0$ *for all* $x\in S^2$, *then the limit map* Φ_∞ *of* S^2 *into the unit 2-sphere in* $\mathfrak{su}(2)$, $\Phi_\infty(x)=\lim_{R\to\infty}\Phi(Rx)$, $x\in S^2$, *is a degree k harmonic map* (*k is the charge of* (A,Φ)).

$SU(2)$ Yang-Mills-Higgs field solutions of charge 1 are obtained in an explicit way as BPS monopoles: $A(x)=(1/\sinh r-1/r)(\partial/\partial r\times e)\cdot dx$, $\Phi(x)=\mp(1/\tanh r-1/r)(\partial/\partial r\cdot e)$, $r=r(x)$([7], [8]). In this BPS monopole case the limit map $\Phi_\infty: S^2\to S^2$ becomes the identity map so that it is automatically holomorphic and hence harmonic.

In Theorem 1 we are able to replace the group $SU(2)$ by an arbitrary compact simple group G (with Lie algebra $\mathfrak{g}$).

Let (A,Φ) be a G Yang-Mills-Higgs field of finite action $\mathcal{Y}(A,\Phi)$ with asymptotical condition $\sup_{|x|\le R}||\Phi|(x)-1|\to 0(R\to\infty)$. Assume the image of the limit map $\Phi_\infty: S^2\to S^N$ ($N=\dim\mathfrak{g}-1$) lies in some orbit $\alpha\subset\mathfrak{g}$ through $X\in S^N$ ([1], [8]). The orbit α of the adjoint action is written as G/K for the isotropy subgroup K at X and carries a homogeneous Kähler space structure imbedded in $\mathfrak{g}$ with the minus Killing form.

Theorem 2. *The limit map* $\Phi_\infty: S^2\to G/K$, *representing the orbit* α, *is a harmonic map, provided that* $[[A(Rx),[A(Rx),\Phi(Rx)]],\Phi(Rx)]=o(1/R^2)(R\to\infty)$ *for a representative* (A,Φ) *in radial gauge of the gauge equivalence class.*

A map Ψ being harmonic from S^2 into a submanifold M of the Euclidean space is characterized by that the Laplacian $\Delta\Psi$ of Ψ is normal to M at any $\Psi(x)$, $x\in S^2$. The above theorems are derived by making use of this fact.

We have moreover similar argument linking magnetic monopoles and holomorphic maps which corresponds just to Donaldson's correspondence.

Theorem 3. *Let* $[(A,\Phi)]$ *be a gauge equivalence class of magnetic monopole with gauge group G of finite action. Assume the limit map* Φ_∞ *lies in an orbit* α. *If some representative* (A,Φ) *of* $[(A,\Phi)]$ *satisfies asymptotical condition* $[A(Rx),\Phi(Rx)]=o(1/R)(R\to\infty)$, *then the map* Φ_∞ *into a homogeneous Kähler space representing the orbit* α *is holomorphic.*

The detailed discussion on these theorems will be given in forthcoming papers.

References

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