28. The Behaviour near the Characteristic Surface of Singular Solutions of Linear Partial Differential Equations in the Complex Domain

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Let $L(z,\partial_z)$ be a linear partial differential operator with the order $m\geq 1$. Its coefficients are holomorphic in a neighbourhood of the origin $z=0$ in C^{n+1}. K is a nonsingular complex hypersurface through $z=0$. In the present paper we treat the equation

$$L(z,\partial_z)u(z)=f(z). \tag{0.1}$$

We assume K is characteristic for $L(z,\partial_z)$. The functions $u(z)$ and $f(z)$ in (0.1) are holomorphic except on K. The results are the following: If $u(z)$ has some growth order near K and the behaviour of $f(z)$ near K is mild, then that of $u(z)$ is also the same type. (Theorems 2.1 and 2.3 and Corollaries). The proofs will be given elsewhere.

§ 1. Definitions. In order to state the results we give notations and definitions: $z=(z_0,z_1,\cdots,z_n)=(z_0,z')$ is the coordinate of C^{n+1}. $|z|=\max\{|z_i|;\,0\leq i\leq n\}$. $\partial_z=(\partial_0,\partial_1,\cdots,\partial_n)=(\partial_0,\partial')$, $\partial_i=\partial/\partial z_i$. We choose the coordinate so that $K=\{z_0=0\}$. We can write the operator $L(z,\partial_z)$ in the form

$$\begin{cases} L(z,\partial_z)=\sum_{k=0}^{m}L_k(z,\partial_z), \\ L_k(z,\partial_z)=\sum_{l=s_k}^{k}A_{k,l}(z,\partial')(\partial_0)^{k-l}, \\ A_{k,l}(z,\partial')=(z_0)^j a_{k,l}(z,\partial') \quad j=j(k,l), \end{cases} \tag{1.1}$$

where $L_k(z,\partial_z)$ is the homogeneous part of order k, $A_{k,s_k}(z,\partial')\not\equiv 0$ if $L_k(z,\partial_z)\not\equiv 0$ and $a_{k,l}(0,z',\partial')\not\equiv 0$ if $A_{k,l}(z,\partial')\not\equiv 0$. We put $s_k=+\infty$ if $L_k(z,\partial_z)\equiv 0$, and $j=j(k,l)=+\infty$ if $A_{k,l}(z,\partial')\equiv 0$.

Let us define the characteristic indices introduced in Ōuchi [7] and [8]. Put $d_{k,l}=l+j(k,l)$ and

$$d_k=\min\{d_{k,l};\,s_k\leq l\leq k\}. \tag{1.2}$$

Put $A=\{(k,d_k)\in R^2:\ 0\leq k\leq m,\ d_k\neq+\infty\}$. Let $\hat{A}$ be the convex hull of A. Let Σ be the lower convex part of the boundary of $\hat{A}$, and Δ be the set of vertices of Σ, $\Delta=\{(k_i,d_{k_i});\,i=0,1,\cdots,l'\}$, $m=k_0>k_1>\cdots>k_{l'}\geq 0$. We put

$$\sigma_i=\max\{1,(d_{k_{i-1}}-d_{k_i})/(k_{i-1}-k_i)\}. \tag{1.3}$$

Then there exists a $p\in N$ such that $\sigma_1>\sigma_2>\cdots>\sigma_{p-1}>\sigma_p=1$. We call $\{\sigma_i;\,1\leq i\leq p\}$ the characteristic indices of $L(z,\partial_z)$ for the surface K.

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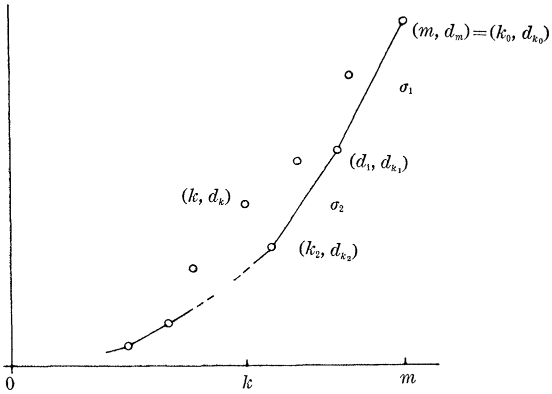
*) Dedicated to Professor Tosifusa KIMURA on his 60th birthday.



For $\Omega^0=\{z_0\in \boldsymbol{C}^1;\ |z_0|\leq R\}$ and $\Omega'=\{z'\in \boldsymbol{C}^n;\ |z'|<R\}$ we put $\Omega=\Omega^0\times\Omega'$, $\Omega^0_\theta=\{z_0\in \boldsymbol{C}^1-\{0\};\ |z_0|\leq R, |\arg z_0|<\theta\}$ and $\Omega_\theta=\Omega^0_\theta\times\Omega'$. For any θ' $(0<\theta'<\theta)$, and any compact set $D\subset\Omega'$, we put $\Omega(\theta', D)=\Omega^0_{\theta'}\times D\subset\Omega_\theta$. Let us define function spaces:

$\mathcal{O}(\Omega)$ is the set of all holomorphic functions on Ω.

$\tilde{\mathcal{O}}(\Omega-K)$ is the set of all holomorphic functions on the universal covering space of $\Omega-K$.

$\mathscr{M}(\Omega-K)=\{f(z)\in\tilde{\mathcal{O}}(\Omega-K);\ f(z)=a(z)\log(z_0)+b(z)/z_0^k,\ a(z),\ b(z)\in \mathcal{O}(\Omega),\ k\in \boldsymbol{N}\}$. The singularities of $f(z)\in\mathscr{M}(\Omega-K)$ are polar or logarithmic.

$\tilde{\mathcal{O}}(\Omega_\theta)$ is the set of all holomorphic functions on Ω_θ.

$\mathrm{Asy}_{\{\gamma\}}(\Omega_\theta)=\{f(z)\in\tilde{\mathcal{O}}(\Omega_\theta)$; For any $\Omega(\theta', D)$ there exist constants $A=A(\theta', D)$ and $B=B(\theta', D)$ such that

(1. 4) $$|f(z)-\textstyle\sum_{k=0}^{N-1}a_k(z')z_0^k|\leq AB^N\Gamma(N/\gamma+1)|z_0|^N \quad \text{in } \Omega(\theta', D),$$

where $a_k(z')\in\mathcal{O}(\Omega')$ $(k=0, 1, \cdots)\}$.

$\mathscr{M}-\mathrm{Asy}_{\{\gamma\}}(\Omega_\theta)=\{f(z)\in\tilde{\mathcal{O}}(\Omega_\theta)$; For any $\Omega(\theta', D)$ there exist constants $A=A(\theta', D)$ and $B=B(\theta', D)$ such that

(1. 5) $$|f(z)-(\textstyle\sum_{k=0}^{N-1}a_k(z')z_0^k)\log(z_0)-\sum_{k=0}^{N-1}b_k(z')z_0^k| \leq AB^N\Gamma(N/\gamma+1)|z_0|^N|\log(z_0)|$$

and

(1. 6) $$|f(z)-(\textstyle\sum_{k=0}^{N}a_k(z')z_0^k)\log(z_0)-\sum_{k=0}^{N-1}b_k(z')z_0^k| \leq AB^N\Gamma(N/\gamma+1)|z_0|^N \quad \text{in } \Omega(\theta', D),$$

where $a_k(z')$, $b_k(z')\in\mathcal{O}(\Omega')$ $(k=0, 1, \cdots)\}$.

$\tilde{\mathcal{O}}_{(\gamma)}(\Omega_\theta)=\{f(z)\in\tilde{\mathcal{O}}(\Omega_\theta)$; For any $\varepsilon>0$ and $\Omega(\theta', D)$ there exists a constant $C_\varepsilon=C(\varepsilon, \theta', D)$ such that

(1. 7) $$|f(z)|\leq C_\varepsilon \exp(\varepsilon|z_0|^{-\gamma}) \quad \text{in } \Omega(\theta', D)\}.$$

§ 2. The behaviour of sulutions. Now let us put

(2. 1) $$\gamma=\sigma_{p-1}-1.$$

Theorem 2.1. *Suppose that* $L(z, \partial_z)$ *satisfies the conditions*

(2. 2) (a) $\sigma_1>1$, (b) $d_{k_{p-1}}=0$, (c) $d_m=s_m$.

Let $u(z)\in\tilde{\mathcal{O}}(\Omega_\theta)$ *be a solution of*

(2. 3) $$L(z, \partial_z)u(z)=f(z)\in\mathrm{Asy}_{\{\kappa\}}(\Omega_\theta) \quad (\kappa\leq\gamma).$$

If $u(z)\in\tilde{\mathcal{O}}_{(\gamma)}(\Omega_\theta)$, *then* $u(z)$ *also belongs to* $\mathrm{Asy}_{\{\kappa\}}(\Omega_\theta)$.

Corollary 2.2. *In Theorem* 2.1, *if* $f(z) \in \mathcal{O}(\Omega)$ *and* $\theta > (\pi/2\gamma) + \pi$, *then* $u(z)$ *is holomorphic in* Ω, *that is,* $u(z)$ *has the holomorphic prolongation to* K.

Theorem 2.3. *Suppose that* $L(z, \partial_z)$ *satisfies the conditions* (2.2). *Let* $u(z) \in \tilde{\mathcal{O}}(\Omega_\theta)$ *be a solution of*

$$(2.4) \qquad L(z, \partial_z)u(z) = f(z) \in \tilde{\mathcal{M}} - \mathrm{Asy}_{\{\kappa\}}(\Omega_\theta) \quad (\kappa \leq \gamma).$$

If $u(z) \in \tilde{\mathcal{O}}_{(\gamma)}(\Omega_\theta)$, *then* $u(z)$ *also belongs to* $\tilde{\mathcal{M}} - \mathrm{Asy}_{\{\kappa\}}(\Omega_\theta)$.

Corollary 2.4. *In Theorem* 2.3, *if* $f(z) \in \tilde{\mathcal{M}}(\Omega - K)$ and $\theta > (\pi/2\gamma) + 2\pi$, *then* $u(z)$ *is in* $\tilde{\mathcal{M}}(\Omega - K)$, *that is,* $u(z)$ *has at most polar or logarithmic singularities on* K.

Let us show examples. Let $L(z, \partial_z)$ be an operator of the form

$$(2.5) \qquad L(z, \partial_z) = (\partial_0)^k + A_m(z, \partial'),$$

where ord. $A_m(z, \partial') = m > k$ and $A_m(0, z', \partial') \not\equiv 0$. Then $\sigma_1 = m/m-k$, $\sigma_2 = 0$ and $\gamma = k/m-k$. The conditions (a), (b), (c) in (2.2) are satisfied. Another example is

$$(2.6) \qquad L(z, \partial_z) = a(z)(\partial_0)^{k_2} + (z_0)^{j_1} a_{k_1, l_1}(z, \partial')(\partial_0)^{k_1 - l_1} + a_{k_0, l_0}(z, \partial')(\partial_0)^{k_0 - l_0},$$

where $\{a(z) a_{k_1, l_1}(z, \partial') a_{k_0, l_0}(z, \partial')\}|_{z_0 = 0} \not\equiv 0$, $k_0 > k_1 > k_2$ and $k_2 > k_0 - l_0$. If $(l_0 - (l_1 + j_1))/(k_0 - k_1) > (l_1 + j_1)/(k_1 - k_2) > 1$, then $\sigma_1 = (l_0 - (l_1 + j_1))/(k_0 - k_1) > \sigma_2 = (l_1 + j_1)/(k_1 - k_2) > \sigma_3 = 1$ and $\gamma = \sigma_2 - 1$. If $(l_0/(k_0 - k_2)) \geqq (l_0 - (l_1 + j_1))/(k_0 - k_1)$, then $\sigma_1 = (l_0/(k_0 - k_2)) > \sigma_2 = 1$ and $\gamma = \sigma_1 - 1$.

Remark 2.5. Corollary 2.2 is a generalization of Theorem 2.1 in [10]. In [10] the conditions for the operator $L(z, \partial)$ are superflous.

Remark 2.6. As for the existence of solutions with singularities on K was investigated in [1], [2], [3], [4], [9], [11] and [12]. The behaviours of singular solutions $u(z)$ near K were investigated in [5] and [6] under the condition that the traces of $u(z)$ on the surface which is transversal to K, say S, are polar on $S \cap K$.

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