29. Deforming Twist Spun 2-Bridge Knots of Genus One

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We work in the *PL* category. Zeeman's k-twist spin of an n-knot K, $k\neq 0$, is a fibered $(n+1)$-knot with fiber punctured k-fold branched cyclic cover of S^{n+2} branched over K [11]. Combining an untwisted deformation of an n-knot with k-twist spinning, $k\neq 0$, Litherland [4] constructed a new fibered $(n+1)$-knot; especially he identified the fiber of an l-roll k-twist spun knot. A 2-bridge knot of genus one $C(2m, 2n)$ has a period q of order 2, that is, rotation q of S^3 with period 2 and axis J which leaves $C(2m, 2n)$ invariant. See Fig. 1, where m (resp. n) denotes the number of half twists, right handed if $m>0$ (resp. $n<0$), left if $m<0$ (resp. $n>0$); $C(4, 6)$ in illustration. Making use of this period, we can construct a deforming twist spun 2-knot. We visualize the fiber (theorem), using the surgery technique by Rolfsen [8]. From this we have:

Corollary. *There exists a fibered* 2-*knot in* S^4 *whose fiber is a punctured Seifert manifold with invariant* $(b; (2,1), (2,1), (2,1))$, $b=1, 4$, *that is, a prism manifold* [6] *with fundamental group* $Q\times Z_{|2b+3|}$, *where* Q *is the quaternion group of order* 8.

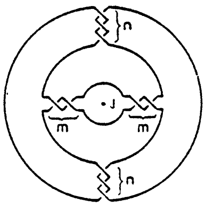

Fig. 1

Hillman [1] determined all the 2-knot groups with finite commutator subgroups. Yoshikawa [10] realized them as twist spun knots in S^4 except in the case when the commutator subgroup is $Q\times Z_m$, m (>1) is odd, when any twist spun knot cannot realize [2, Chapter 5] and Yoshikawa only got a fibered 2-knot in a homotopy 4-sphere. Morichi [5] realized an embedding of every punctured prism manifold in S^4. Plotnick and Suciu [7] determined all the fibered 2-knots in a homotopy 4-sphere with fiber a punctured spherical space form; it is not known weather all of them, including the above case, can be realized as fibered 2-knots in S^4.

Construction of a fibered 2-knot. The circle S^1 is taken to be the quotient space either

$$R/\theta\sim\theta+1 \qquad \text{for all } \theta\in R, \text{ or } I/0\sim 1,$$

where I is the unit interval $[0, 1]$.

In the quotient space

$$S^3 \cong S^1 * S^1 = S^1 \times S^1 \times I/\sim$$

with the identification

$$(x, y, 0) \sim (x, y', 0) \qquad \text{for all } x, y, y' \in S^1,$$
$$(x, y, 1) \sim (x', y, 1) \qquad \text{for all } x, x', y \in S^1,$$

let V_0, V_1, V_1' be the solid tori $S^1 \times S^1 \times [0, 1/2]/\sim$, $S^1 \times S^1 \times [1/2, 1]/\sim$, $S^1 \times S^1 \times [3/4, 1]/\sim$, respectively; so $S^3 \cong V_0 \cup V_1$. Let $C_i = S^1 \times S^1 \times \{i\}/\sim$ be the core of V_i, $i=0, 1$. Define the homeomorphisms $f_s, g_s, h_s : S^3 \to S^3$, $s \in R$, by

$$f_s(x, y, t) = (x, y+s, t);$$

$$g_s(x, y, t) = \begin{cases} (x, y+s, t) & \text{if } 0 \leqq t < 1/2, \\ (x, y+(3-4t)s, t) & \text{if } 1/2 \leqq t < 3/4, \\ (x, y, t) & \text{if } 3/4 \leqq t \leqq 1; \end{cases}$$

$$h_s(x, y, t) = \begin{cases} (x+s, y, t) & \text{if } 0 \leqq t < 1/2, \\ (x+(3-4t)s, y, t) & \text{if } 1/2 \leqq t < 3/4, \\ (x, y, t) & \text{if } 3/4 \leqq t \leqq 1; \end{cases}$$

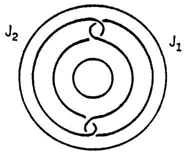

Fig. 2

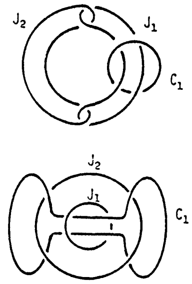

Fig. 3

Let J_1 and J_2 be the simple closed curves in *int* V_0 as shown in Fig. 2 such that $g_{1/2}(J_i) = J_i$, $i=1, 2$. Let V_0^* be the result of a Dehn surgery on $J_1 \cup J_2$ in V_0 with surgery coefficients $1/n_1$ and $1/n_2$, $n_i \in Z$, that is, V_0^* is the manifold obtained from V_0 by removing a tubular neighborhood N_i of J_i and sewing it back by means of a homeomorphism $N_i \to N_i$, which takes a meridian to (meridian)$+n_i$ (longitude); cf. [8, Chapter 9]. Then $V_0^* \cup V_1$ is again a 3-sphere Σ^3, and C_1 is a 2-bridge knot $C(-2n_1, 2n_2)$ in Σ^3. See Fig. 3, where $J_1 \cup J_2 \cup C_1$ constitutes the Borromean ring, see e.g., [8, 3F4]. Let $X = \Sigma^3$-*int* V_1^*, the exterior of $C(-2n_1, 2n_2)$. Let f_s^*, g_s^*, h_s^* be the homeomorphisms $\Sigma^3 \to \Sigma^3$ induced by f_s, g_s, h_s, respectively. Then $f_{1/2}^*$ is just the period q given in the introduction. Let $p : X \to S^1$ be a map extending the projection

$$\partial X = \partial V_1' = S^1 \times S^1 \times \{3/4\}/\sim \ni (x, y, 3/4) \to x \in S^1.$$

Cf. [4, Definition 2.3]. Note that $p(g_{b/2}^* | X) = p$, $b \in Z$. Let $B_- = S^1 \times [0, 1/2] \times [3/4, 1]/\sim$ and $C_- = C_1 \cap B_-$. Then (B_-, C_-) is a standard ball pair. Using the complementary ball pair (B_+, C_+) in (Σ^3, C_1) we construct a deforming twist spun 2-knot:

$$(S^4, K_{a,b}) = \partial(B_+, C_+) \times B^2 \cup_\beta (B_+ \times \partial B^2, \cup_{s-I} (\phi_s(C_+), s)),$$

where $\phi_s = h^*_{as} g^*_{bs/2}$, $a, b \in Z$. Let $M = V'_1 \cup \{(v, t) \in X \times S^1 / g^*_{b/2} \mid p(v) = at\}$, where $X \times S^1 / g^*_{b/2}$ is the mapping torus $X \times I / [(v, 0) \sim (g^*_{b/2}(v), 1)\,;\, v \in X]$ and $\beta : \partial V'_1 \to \{(v, t) \in \partial X \times S^1 / g^*_{b/2} \mid p(v) = at\}$, $\beta(x, y, 3/4) = ((ax, y, 3/4), x)$. By [4, Section 4] if $a \neq 0$, then $K_{a,b}$ is a fibered knot with fiber *punc* $M = M - \{pt\}$ with characteristic map given by

$$(v, t) \to (v, t + 1/a) \quad \text{on} \quad \{(v, t) \in X \times S^1 / g^*_{b/2} \mid p(v) = at\}$$

If b is even, then $K_{a,b}$ is a $b/2$-roll, a-twist spin of $C(-2n_1, 2n_2)$; cf. [4, Corollary 5.2]. Thus we have

Theorem. *Let M be the result of a Dehn surgery on S^3 along the $2a$-component link $L = J^1_1 \cup J^1_2 \cup J^2_1 \cup J^2_2 \cup \cdots \cup J^a_1 \cup J^a_2$ as shown in Fig. 4, where b denotes the number of half twists, left handed if $b > 0$, and right if $b < 0$, with coefficients $1/n_i$ for J^k_i, $i = 1, 2$, $k = 1, 2 \cdots, a$. Then if $a > 0$, $K_{a,b}$ is a fibered 2-knot with fiber punc M.*

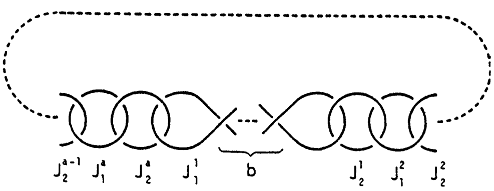

Fig. 4

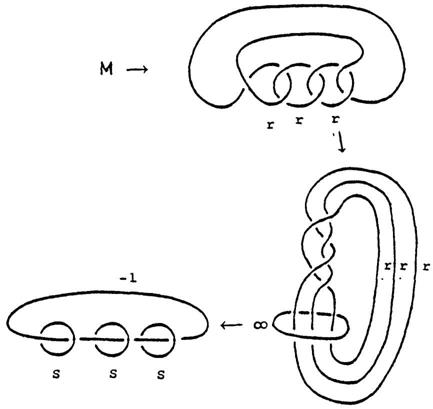

Fig. 5

Proof of Corollary. In theorem, let $n_1 = 1$, $n_2 = -1$, $b = -1$, $a = 3$ (resp. $n_1 = -3$, $n_2 = -1$, $b = -1$, $a = 3$). Then M is the prism manifold as shown in Fig. 5, where $r = 3$ (resp. $5/3$), $s = 2$ (resp. $2/3$).

Example 1. The right-handed trefoil $C(-2, 2)$. Since the trefoil knot is the torus knot of type $(2, 3)$, $K_{a,b}$ is just the $(a - 3b)$-twist spun trefoil; $K_{a,b} = K_{a-3b,0}$ [4, Corollary 6.4]. For the left-handed trefoil $C(2, -2)$, $K_{a,b} = K_{a+3b}$. We give in Fig. 6 the fibers of $K_{1,1}$, $K_{1,-1}$, $K_{3,1}$, which coincide with those of $K_{-2,0}$, $K_{4,0}$, $K_{0,0}$ [8, 10D].

Example 2. $C(2, 2n)$. The fibers of $K_{1,-1}$ and $K_{2,-1}$ are the lens space $L(4n+1, n)$ and the connected sum of the lens spaces $L(2n+1, n) \# L(2n+1, n)$ (Fig. 7), so $K_{1,-1}$ is the 2-twist spun 2-bridge knot and $K_{2,-1}$ is the 2-cable about the 2-twist spun 2-bridge knot in the sense of [3] considering the characteristic map [9].

Example 3. $C(2n, 2n)$. The knot is amphicheiral. It is easy to see that the fiber of $K_{a,b}$ and $K_{a,-b}$ are homeomorphic by an orientation reversing homeomorphism.

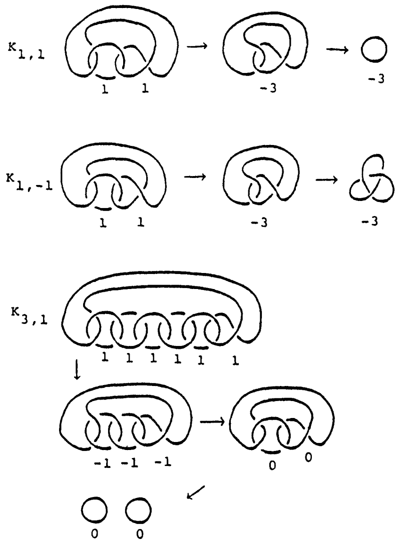

Fig. 6

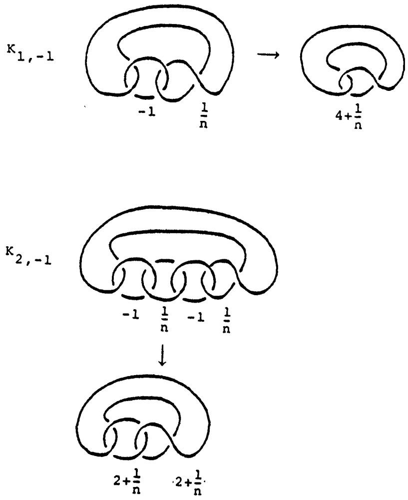

Fig. 7

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