28. *On the Essential Self-adjointness of Pseudo-differential Operators*

By Michihiro NAGASE*) and Tomio UMEDA**)

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§1. **Introduction.** In a rigorous treatment of quantum mechanics it is basically important to consider the problem: Is a quantum Hamiltonian self-adjoint? In the present paper we state several theorems on the essential self-adjointness of pseudo-differential operators with Weyl symbols. Applying the theorems we can show the essential self-adjointness of Weyl quantized Hamiltonians.

In [8], M. A. Shubin gives a proof of essential self-adjointness of pseudo-differential operators by using a global hypo-elliptic estimate. However, we can obtain the theorems without use of hypo-ellipticity. In order to get our main result we use an algebra of spatially inhomogeneous pseudo-differential operators, which are studied, for example, in [1], [3] and [4].

We do not give detailed proofs of the theorems here. The detailed proofs will be published elsewhere.

§2. **An algebra of pseudo-differential operators.** We give here some results on pseudo-differential operators. The results have already been obtained fundamentally by Iwasaki [3] and Kumano-go and Taniguchi [4], however, we have to reproduce some of their results in a suitable form to our purpose.

Definition 2.1 (see [3] and [4]). A smooth function $\lambda(x,\xi)$ on $\boldsymbol{R}^d\times\boldsymbol{R}^d$ is called a basic weight function if

(1) $1\leq\lambda(x+y,\xi)\leq C_0\langle y\rangle^{\tau}\lambda(x,\xi)$,

(2) $|\lambda_{(\beta)}^{(\alpha)}(x,\xi)|\leq C_{\alpha\beta}\lambda(x,\xi)^{1-|\alpha|+\delta|\beta|}$ for any α and β,

where τ and δ are non-negative constants with $0\leq\delta<1$, $\langle y\rangle=(1+|y|^2)^{1/2}$ and

$$\lambda_{(\beta)}^{(\alpha)}(x,\xi)=\left(-i\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\lambda(x,\xi).$$

Definition 2.2 (see [3] and [4]). Let m, δ and ρ be real numbers with $0\leq\delta<\rho\leq1$. We say that a smooth function $p(x,\xi)$ belongs to the class $S_{\lambda,\rho,\delta}^m$, if $p(x,\xi)$ satisfies

$$|p_{(\beta)}^{(\alpha)}(x,\xi)|\leq C_{\alpha\beta}\lambda(x,y)^{m-\rho|\alpha|+\delta|\beta|}\qquad\text{for any }\alpha\text{ and }\beta.$$

Let $\mathcal{S}$ denote the Schwartz space of rapidly decreasing functions on $\boldsymbol{R}^d$. For $p(x,\xi)\in S_{\lambda,\rho,\delta}^m$ we define operators $p(X,D)$ and $p^w(X,D)$ on $\mathcal{S}$ by

$$p(X,D)u(x)=(2\pi)^{-d}\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)d\xi,$$

*) Department of Mathematics, College of General Education, Osaka University.
**) Department of Mathematics, Faculty of Science, Osaka University.

$$p^w(X, D)u(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} p\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

Then we have the following theorems.

Theorem 2.1. (i) *Operators $p(X, D)$ and $p^w(X, D)$ are continuous from $\mathcal{S}$ to $\mathcal{S}$. If $p(x, \xi)$ is real-valued, then the operator $p^w(X, D)$ is symmetric on $\mathcal{S}$: $(p^w(X, D)u, v) = (u, p^w(X, D)v)$, where $(\cdot, \cdot)$ denotes the inner product on $L^2(\boldsymbol{R}^d)$.*

(ii) *If $p(x, \xi) \in S^0_{\lambda,\rho,\delta}$, then we have*

$$\|p(X, D)u\| \leq C|p|_l \|u\|$$
$$\|p^w(X, D)u\| \leq C|p|_l \|u\|,$$

where $\|\cdot\|$ denotes the norm on $L^2(\boldsymbol{R}^d)$,

$$|p|_l = \max_{|\alpha|+|\beta|\leq l} \sup_{(x,\xi)} \{|p^{(\alpha)}_{(\beta)}(x, \xi)| \lambda(x, \xi)^{\rho|\alpha|-\delta|\beta|}\},$$

and l is a sufficiently large integer.

Theorem 2.2. (i) *If $p_k(x, \xi) \in S^{m_k}_{\lambda,\rho,\delta}$ $(k=1, 2)$, then there exists a symbol $p(x, \xi) \in S^{m_1+m_2}_{\lambda,\rho,\delta}$ such that*

$$p_1(X, D)p_2(X, D)u(x) = p(X, D)u(x) \qquad \text{for } u \in \mathcal{S},$$

and we have the asymptotic expansion

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p^j(x, \xi),$$

where

$$p^j(x, \xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!} p_1^{(\alpha)}(x, \xi) p_{2(\alpha)}(x, \xi).$$

(ii) *If $p(x, \xi) \in S^m_{\lambda,\rho,\delta}$, then there exists a symbol $\tilde{p}(x, \xi) \in S^m_{\lambda,\rho,\delta}$ such that*

$$p^w(X, D)u(x) = \tilde{p}(X, D)u(x) \qquad \text{for } u \in \mathcal{S},$$

and we have the asymptotic expansion

$$\tilde{p}(x, \xi) \sim \sum_{j=0}^{\infty} p^j(x, \xi),$$

where

$$p^j(x, \xi) = \left(\frac{1}{2}\right)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} p^{(\alpha)}_{(\alpha)}(x, \xi).$$

§3. Essential self-adjointness of pseudo-differential operators. In this section we assume that $m>0$, $0 \leq \delta < \rho \leq 1$ and we consider a real valued symbol $p(x, \xi)$ in $S^m_{\lambda,\rho,\delta}$. We work in the Hilbert space $L^2(\boldsymbol{R}^d)$.

Theorem 3.1. *If $p(x, \xi)$ satisfies*

$$(3.1) \qquad |p^{(\alpha)}_{(\beta)}(x, \xi)| \leq C_{\alpha\beta} |p(x, \xi) + i| \lambda(x, \xi)^{-\rho|\alpha|+\delta|\beta|}$$

for any α and β, then the operator $p^w(X, D)$ is essentially self-adjoint on $\mathcal{S}$.

A sketch of the proof. We have only to show that the sets $\{p^w(X, D) \pm i\mu\}(\mathcal{S})$ are dense in $L^2(\boldsymbol{R}^d)$ for some positive constant $\mu \neq 0$ (see [7]). We prove this by constructing the symbols $q^\pm_\mu(x, \xi) \in S^0_{\lambda,\rho,\delta}$ which satisfy

$$(3.2) \qquad \{p^w(X, D) \pm i\mu\} \cdot q^\pm_\mu(X, D)u(x) = u(x) + r^\pm_\mu(X, D)u(x) \qquad \text{for } u \in \mathcal{S},$$

where $r^\pm_\mu(X, D)$ are bounded operators with

$$(3.3) \qquad \|r^\pm_\mu(X, D)u\| \leq \frac{C}{\mu} \|u\| \qquad \text{for } u \in \mathcal{S}.$$

We can construct the $q_\mu^\pm(x,\xi)$ by using the theorems in Section 2. It follows from (3.2) and (3.3) that $\{p^w(X,D)\pm i\mu\}\cdot q_\mu^\pm(X,D)$ are bounded operators on $L^2(\boldsymbol{R}^d)$ with bounded inverses. Since $q_\mu^\pm(X,D)$ are continuous from $\mathcal{S}$ to $\mathcal{S}$, we can see that the sets $\{p^w(X,D)\pm i\mu\}(\mathcal{S})$ are dense in $L^2(\boldsymbol{R}^d)$ for sufficiently large $\mu>0$. Q.E.D.

The following theorem is an easy consequence of Theorem 3.1.

Theorem 3.2. *If $p(x,\xi)$ satisfies*

$$p(x,\xi)^2+\mu_0^2\geq c_0\lambda(x,\xi)^{2m}$$

for some constants μ_0 and $c_0>0$, then the operator $p^w(X,D)$ is essentially self-adjoint on $\mathcal{S}$.

By a similar method to the proof of Theorem 3.1, we can obtain the following result.

Theorem 3.3. *Assume that $p(x,\xi)$ is non-negative and satisfies* (3.1). *If a smooth function $V(x)$ is real valued and satisfies that $V(x)+\mu_1\geq 1$, $|V(x)|\leq C\langle x\rangle^M$ and*

$$|\partial^\alpha V(x)|\leq C_\alpha\{V(x)+\mu_1\}$$

for some constants μ_1 and M, then the operator $p^w(X,D)+V(x)$ is essentially self-adjoint on $\mathcal{S}$.

It is desirable to show that the operators are essentially self-adjoint on $C_0^\infty(\boldsymbol{R}^d)$. To do so, we need a lemma.

Lemma 3.1. *There exist positive constants C and s such that*

$$\|p(X,D)u\|\leq C\|u\|_{m,s} \qquad \text{for } u\in\mathcal{S},$$

where $\|u\|_{m,s}=\|\langle D\rangle^m\langle x\rangle^s u\|$.

Using this lemma we have

Theorem 3.4. *If $p^w(X,D)$ is essentially self-adjoint on $\mathcal{S}$, then $p^w(X,D)$ is essentially self-adjoint on $C_0^\infty(\boldsymbol{R}^d)$.*

From this theorem we can deduce the variants of Theorems 3.1, 3.2 and 3.3.

Examples (see [2], [5] and [6]). Let $a(x)=(a_1(x),\cdots,a_d(x))$ satisfy $|\partial^\alpha a_j(x)|\leq C_\alpha$ for any $\alpha\neq 0$. Put $p_\nu(x,\xi)=\{|\xi-a(x)|^2+\nu^2\}^{1/2}$ $(\nu>0)$. Then taking $\lambda(x,\xi)=\{|\xi-a(x)|^2+1\}^{1/2}$ we can see that the symbol $p_\nu(x,\xi)$ belongs to $S^1_{\lambda,1,0}$ and satisfies the assumption of Theorem 3.2. Hence $p_\nu^w(X,D)$ is essentially self-adjoint on $\mathcal{S}$. Thus, by Theorem 3.4, $p_\nu^w(X,D)$ is essentially self-adjoint on $C_0^\infty(\boldsymbol{R}^d)$. Furthermore, if $V(x)$ is a smooth function satisfying the assumption of Theorem 3.3, then $p_\nu^w(X,D)+V(x)$ is essentially self-adjoint on $\mathcal{S}$, and consequently essentially self-adjoint on $C_0^\infty(\boldsymbol{R}^d)$.

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