3. On the First Eigenvalue of Some Quasilinear Elliptic Equations

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1. Introduction. Let Ω be a bounded domain in $\boldsymbol{R}^N$ with smooth boundary $\partial\Omega$. For given $p \in (1, +\infty)$, $a \in L^\infty_+(\Omega) = \{f \in L^\infty(\Omega)\,;\ f(x) \geq 0$ a.e. $x \in \Omega\}$ and $b \in L^\infty_0(\Omega) = \{f \in L^\infty(\Omega)\,;\ f^+(\cdot) = \max(f(\cdot), 0) \not\equiv 0\}$, we consider the following eigenvalue problem:

$$(E)_\lambda \quad \begin{cases} (1) \quad -\Delta_p u(x) + a(x)|u|^{p-2}u(x) = \lambda b(x)|u|^{p-2}u(x), \ x \in \Omega, \ \lambda > 0, \\ (2) \quad u(x) = 0, \ x \in \partial\Omega, \end{cases}$$

where $\Delta_p u(x) = \operatorname{div}(|\nabla u|^{p-2}\nabla u(x))$.

The main purpose of this paper is to show that there exists a positive number λ_1, *the first eigenvalue*, such that $(E)_\lambda$ admits a positive solution if and only if $\lambda = \lambda_1$ and that λ_1 is simple, i.e., solutions of $(E)_{\lambda_1}$ forms a one dimensional subspace of $W^{1,p}_0(\Omega)$. Here u is said to be a solution of $(E)_\lambda$ if u belongs to $W^{1,p}_0(\Omega)$ and satisfies (1) in the sense of distribution. For the case where $a \equiv 0$ and $b \equiv 1$, the simplicity of λ_1 has been shown under some additional assumptions. When $N = 1$, it is shown in [2] that all eigenvalues λ_k $(k \in \boldsymbol{N})$ are simple and that all eigenfunctions u_k associated with λ_k have $(k-1)$ isolated zeros in Ω. If Ω is a ball, DeThélin [5] showed the simplicity of λ_1 in the class of radially symmetric solutions by using the theory of rearrangement. Recently, Sakaguchi [4] made an argument based on a strong maximum principle to prove that λ_1 is simple provided that $\partial\Omega$ is connected. Our method of proof is quite different from those in [2], [4], [5], and requires neither the connectedness of $\partial\Omega$ nor the positivity of $b(\cdot)$.

We define $\lambda_1 = \lambda_1(a, b)$ by

$$(3) \qquad 1/\lambda_1 = \sup\{R(v) := B(v)/A(v)\,;\ v \in W := W^{1,p}_0(\Omega)\backslash\{0\}\},$$

where $A(v) = \int_\Omega \{|\nabla u(x)|^p + a(x)|u(x)|^p\}dx$ and $B(v) = \int_\Omega b(x)|u(x)|^p dx$. Then our main result is stated as follows:

Theorem 1. *Eigenvalue problem $(E)_\lambda$ has a nontrivial nonnegative solution u if and only if $\lambda = \lambda_1$ and $J_{\lambda_1}(u) := A(u) - \lambda_1 B(u) = 0$. Furthermore, the eigenvalue λ_1 is simple, more precisely, the set of all solutions of $(E)_{\lambda_1}$ consists of $\{tu_1\,;\ t \in \boldsymbol{R}^1\}$, where u_1 is a solution of $(E)_{\lambda_1}$ such that $u_1 \in C^{1,\theta}(\bar{\Omega})$ for some $\theta \in (0, 1)$ and $u_1(x) > 0$ for all $x \in \Omega$.*

2. Some lemmas. To prove Theorem 1, we here prepare some lemmas.

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Lemma 2. *Let u be a solution of $(E)_\lambda$. Then $u \in C^{1,\theta}(\bar{\Omega})$ for some $\theta \in (0, 1)$. Furthermore, if $u \geqq 0$ in Ω, then $u > 0$ in Ω.*

Proof. The very same verification as for Theorem 2 of [3] assures that $u \in L^\infty(\Omega)$. Then above assertions follows from Proposition 3.7 of [6] and Theorem 1.1 of [7]. Q.E.D.

Lemma 3. *Let $F(x, u): \Omega \times R^1 \to R^1$ be measurable in x and monotone nondecreasing in u. Let $u_1, u_2 \in W^{1,p}(\Omega)$ satisfy*

$$-\Delta_p u_1(x) + F(x, u_1(x)) \leqq -\Delta_p u_2(x) + F(x, u_2(x)) \quad \text{in } W^{-1,p'}(\Omega), \quad p' = p/(p-1). \tag{4}$$

Then $u_1 \leqq u_2$ on $\partial\Omega$ implies $u_1 \leqq u_2$ in Ω.

Proof. Put $w(x) = \max(u_1(x) - u_2(x), 0)$. Then, by Corollary A.6 of [1], $w \in W_0^{1,p}(\Omega)$. Multiplying (4) by w and using the monotonicity of $F(x, \cdot)$, we find that the integration of $(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)(\nabla u_1 - \nabla u_2)$ over $D = \{x \in \Omega\,;\, u_1(x) \geqq u_2(x)\}$ is non-positive. Since $-\Delta_p$ is strictly monotone, we deduce that $\nabla u_1 = \nabla u_2$ in D, whence follows $\nabla w = 0$, i.e., $u_1 \leqq u_2$ in Ω. Q.E.D.

Lemma 4. *Let $u \in W_0^{1,p}(\Omega) \cap C^1(\bar{\Omega})$ satisfy*

$$\begin{cases} -\Delta_p u(x) + M u^{p-1}(x) \geqq 0 & \text{in } W^{-1,p'}(\Omega),\ M \geqq 0, \\ u > 0 & \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{5}$$

Then the outer normal derivative $\partial u/\partial n$ of u is strictly negative on $\partial\Omega$.

Proof. For every $x_0 \in \partial\Omega$ and a sufficiently small $R > 0$, there exists $y \in \Omega$ such that $B_{2R}(y) \subset \Omega$ and $x_0 \in \partial B_{2R}(y) \cap \partial\Omega$, where $B_\rho(z) = \{x \in \boldsymbol{R}^N\,;\, |z - x| < \rho\}$. Set

$$v(x) = \alpha(3R - r)^\delta - \alpha R^\delta, \quad r = |x - y|, \quad \alpha > 0. \tag{6}$$

Then it is easy to see that α and R (resp. δ) may be chosen small (resp. large) enough so that $-\Delta_p v + M v^{p-1} \leqq 0$ in $\Omega_R = B_{2R}(y) \backslash \overline{B_R(y)}$ and $v \leqq u$ on $\partial\Omega_R$. Hence, from Lemma 3, we deduce that $v(x) - v(x_0) \leqq u(x) - u(x_0)$ for all $x \in \Omega_R$, whence follows the assertion. (See Lemma A.3 of [4].) Q.E.D.

3. Proof of Theorem 1. The proof is devided into five steps.

(i) $0 < \lambda_1 < +\infty$: Suppose that $B(u) \leqq 0$ for all $u \in W_0^{1,p}(\Omega)$. Then, since there exist $v_n \in W_0^{1,p}(\Omega)$ such that $v_n \geqq 0$ and $v_n \to b^+ = \max(b, 0)$ in $L^p(\Omega)$, we obtain $B(b^+) \leqq 0$, which gives the contradiction $b^+ \equiv 0$. Hence there exists an element $u_0 \in W_0^{1,p}(\Omega)$ satisfying $B(u_0) > 0$. Thus $0 < \lambda_1 < 1/R(u_0)$. Furthermore, by multiplying (1) by u and using Hölder's inequality, we can obtain the lower bound of λ_1: $\lambda_1 \geqq (C_q |b^+|_{L^\infty} |\Omega_+|^{(q-p)/q})^{-1}$ for all q such that $C_q := \sup\{|u|_{L^q}/|\nabla u|_{L^p}\,;\, u \in W\} < +\infty$, where $\Omega_+ = \{x \in \Omega\,;\, b(x) > 0\}$.

(ii) *$(E)_\lambda$ has no nontrivial solution for $\lambda \in [0, \lambda_1)$*: Let u be a solution of $(E)_\lambda$. Then multiplication of (1) by u gives $R(u) = 1/\lambda > 1/\lambda_1$, which contradicts (3).

(iii) *u is a solution of $(E)_{\lambda_1}$ if and only if $J_{\lambda_1}(u) = 0$*: The "only if" part can be proved as in step (ii). Let $J_{\lambda_1}(u) = 0$, then (3) implies $J_{\lambda_1}(u) = \min\{J_{\lambda_1}(u)\,;\, u \in W\} = 0$. Hence Fréchet derivative of J_{λ_1} at u vanishes, i.e., u is a solution of $(E)_{\lambda_1}$. Moreover, since $J_\lambda(u) = J_\lambda(|u|)$ and $W_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$, there exists a non-negative function u_1 such that $J_{\lambda_1}(u_1)=0$. Then, by Lemma 2, $(E)_{\lambda_1}$ has always a positive solution $u_1 \in C^{1,\theta}(\bar{\Omega})$.

(iv) *λ_1 is simple*: Let u and v be positive solutions of $(E)_{\lambda_1}$. Then $M(t,x)=\max(u(x), tv(x))$ and $m(t,x)=\min(u(x), tv(x))$ belong to $W_0^{1,p}(\Omega)$ and satisfy $J_{\lambda_1}(M(t,\cdot))+J_{\lambda_1}(m(t,\cdot))=J_{\lambda_1}(u)+J_{\lambda_1}(tv)=0$ (see [1]). Hence, by (3), $J_{\lambda_1}(M(t,\cdot))=J_{\lambda_1}(m(t,\cdot))=0$. Then $M(t,x)$ is a solution of $(E)_{\lambda_1}$, and by Lemma 2 $M(t,\cdot)\in C^{1,\theta}(\bar{\Omega})$ for all $t\geqq 0$. For any $x_0 \in \Omega$, set $t_0=u(x_0)/v(x_0)>0$. Since $u(x_0+he)-u(x_0)\leqq M(t_0, x_0+he)-M(t_0, x_0)$ for all unit vectors e, dividing this inequality by $h>0$ or $h<0$, and letting $h\to\pm 0$, we find $\nabla_x u(x_0)=\nabla_x M(t_0, x_0)$, and similarly $\nabla_x M(t_0, x_0)=t_0\nabla_x(x_0)$. Thus we obtain $\nabla_x(u/v)(x_0)=0$, i.e., $u(x)/v(x)\equiv$Const. in Ω.

(v) *$(E)_\lambda$ has no positive solution for $\lambda>\lambda_1$*: Let u and v be positive solutions of $(E)_{\lambda_1}$ and $(E)_\lambda$ respectively. By virtue of Lemmas 2 and 4, u and v may be choosen so that $u\leqq v$ in Ω. For the time being, assume $b\geqq 0$. Then $-\Delta_p u+au^{p-1}=\lambda_1 bu^{p-1}\leqq\lambda_1 bv^{p-1}=-\Delta_p(\eta v)+a(\eta v)^{p-1}$, where $\eta=(\lambda_1/\lambda)^{1/(p-1)}<1$. Therefore Lemma 3 assures that $u\leqq\eta v$ in Ω. Repeating this procedure, we deduce that $u\leqq\eta^n v$ in Ω for all $n\in N$, whence follows $u\equiv 0$. This is a contradiction. Let $b^+=\max(b,0)$ and $b^-=\max(-b,0)$. Then above results say that the equation $-\Delta_p w+\{a+\lambda b^-\}w^{p-1}=\mu b^+ w^{p-1}$ has a nontrivial positive solution w if and only if $\mu=\mu_1=\lambda_1(a+\lambda b^-, b^+)$ and $I_{\mu_1}(w)=A(w)+\lambda\int_\Omega b^-(x)|w|^p dx-\mu_1\int_\Omega b^+(x)|w|^p dx=\min\{I_{\mu_1}(z); z\in W\}=0$. Since v is a positive solution of the above equation with $\mu_1=\lambda$, we deduce that $\mu_1=\lambda$ and $I_{\mu_1}(v)=I_\lambda(v)=J_\lambda(v)=\min\{J_\lambda(z); z\in W\}=0$. However, $J_\lambda(u)=J_{\lambda_1}(u)-(\lambda-\lambda_1)B(u)<0$. This is a contradiction. Q.E.D.

4. Remark. By the same argument as in [4] with obvious modifications, we can show the following result: "*Let $a\equiv 0$, $b\geqq 0$ and $b\in C^\theta(\Omega)$ for some $\theta\in(0,1)$, and let Ω be convex and $b(\cdot)$ be concave. Then, every positive solution u of $(E)_{\lambda_1}$ is* log-*concave*, i.e., log *u is a concave function.*"

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