23. A Problem on Quadratic Fields

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Let k be a quadratic field, Δ_k the discriminant and M_k the Minkowski constant:

$$M_k=\begin{cases}\dfrac{1}{2}\sqrt{\Delta_k} & \text{if } k \text{ is real,}\\ \dfrac{2}{\pi}\sqrt{-\Delta_k} & \text{if } k \text{ is imaginary.}\end{cases}$$

Consider the finite set of prime numbers

$$\Pi_k=\{p,\ \text{rational prime};\ p\leqq M_k\}.$$

There are exactly 8 fields for which $\Pi_k=\phi$. They make up an exceptional family

$$E_8=\{k=\boldsymbol{Q}(\sqrt{m});\ m=-1,\ \pm 2,\ \pm 3,\ 5,\ -7,\ 13\}.$$

For any k, let χ_k denote the Kronecker character. The character splits Π_k into 3 disjoint parts:

$$\Pi_k^0=\{p\in\Pi_k;\ \chi_k(p)=0\},$$
$$\Pi_k^-=\{p\in\Pi_k;\ \chi_k(p)=-1\},$$
$$\Pi_k^+=\{p\in\Pi_k;\ \chi_k(p)=+1\}.$$

Consider, next, the 3 families of fields:

$$K^0=\{k;\ \Pi_k=\Pi_k^0\},$$
$$K^-=\{k;\ \Pi_k=\Pi_k^-\},$$
$$K^+=\{k;\ \Pi_k=\Pi_k^+\}.$$

The problem is to determine explicitly the 3 families. Since E_8 is common to all 3 families, it is enough to determine K^0-E_8, K^--E_8, K^+-E_8, respectively.

(I) K^0-E_8. This is the easiest part of the problem and one settles it completely. Namely,

(1) $$K^0-E_8=\{k=\boldsymbol{Q}(\sqrt{m});\ m=-5,\ \pm 6, 7, 10, 15, \pm 30\}.$$

Proof of (1). Let p_n denote the n th prime. Using the well-known Chebyshev's inequality, $p_{n+1}<2p_n$, $n\geqq 1$, one proves by induction that

(2) $$p_{n+1}^2<\frac{1}{4}p_1p_2\cdots p_n \qquad \text{when } n\geqq 5.$$

For any $k\in K^0-E_8$, choose n so that $p_n\leqq M_k<p_{n+1}$. Since $p_n\leqq M_k$, we have $p_1\cdots p_n\,|\,\Delta_k$ by definition of K^0, and so

$$p_{n+1}^2>M_k^2\geqq\frac{1}{4}|\Delta_k|\geqq\frac{1}{4}p_1\cdots p_n.$$

Hence, by (2), $n\leqq 4$ and $M_k<p_5=11$, from which one easily verifies (1).

(II) K^--E_8. This part of the problem is almost settled by H. M. Stark

[3] and M. -G. Leu [2]. To be more precise, if k is imaginary, it is easy to see that

(3) $k \in K^{-} \Longleftrightarrow h_k=1$ (h_k denotes the class number of k)*).

Therefore [3] is good enough to handle the imaginary part of $K^{-}-E_8$. As for the real part, [2] determined fields with *at most one exception*. The list of fields is (conjecturally) as follows:

(4) $K^{-}-E_8=\{k=\boldsymbol{Q}(\sqrt{m})\,;\, m=-11, -19, -43, -67, -163, 21, 29, 53, 77, 173, 293, 437\}$.

(III) $K^{+}-E_8$. This part of the problem is closely connected with the problem on the least quadratic non-residue. At present, I do not have a proof for the following (conjectural) list hinted by the machine computation:

(5) $K^{+}-E_8=\{k=\boldsymbol{Q}(\sqrt{m})\,;\, m=-15, -23, -47, -71, -119, 17, 33, 73, 97\}$.

Remark. Needless to say that any number field k has the Minkowski constant M_k. Hence one can create the similar problem for k's which are not quadratic by using the 'character' belonging to k.

References

[1] H. K. Kim, M.-G. Leu, and T. Ono: On two conjectures on real quadratic fields. Proc. Japan Acad., **63**A, 222–224 (1987).

[2] M.-G. Leu: On a conjecture of Ono on real quadratic fields. ibid., **63**A, 323–326 (1987).

[3] H. M. Stark: A complete determination of the complex quadratic fields of class-number one. Michigan Math. J., **14**, 1–27 (1967).

*) If k is real, we have one-way only: $k \in K^{-} \Rightarrow h_k=1$.