22. On Propagation of Regular Singularities for Solutions of Nonlinear Partial Differential Equations. I

By Tan ISHII
Department of Mathematics, Meijigakuin University

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§0. **Introduction.** In this paper we shall study in a complex domain what sort of conditions on a hypersurface are necessary if a solution of a given nonlinear partial differential equation with holomorphic coefficients has regular singularities along this hypersurface and give a simple necessary condition for the equation with polynomial type nonlinearity. As it is well known, for a linear equation such a hypersurface must be a characteristic hypersurface for the operator. Tsuno [5] showed that for quasilinear equations such a hypersurface must be also characteristic for the linear part of the operator if the solution is not too singular. Recently, for semi-linear equations Ishii-Kobayashi [2] determined the infimum of the exponents of regular singularities of solutions which propagate along characteristic hypersurface for the linear part of the equations. Moreover, when the exponent is just equal to the infimum, they constructed a solution which has singularities on an almost any given non-characteristic hypersurface. In the real domain, Bony [1] investigated the propagation of singularities with rather high regularity for quasi-linear equations and showed that is performed along bicharacteristic curves of the linearized characteristic equation microlocally. On the other hand, Kobayashi-Nakamura [4] succeeded in construction of solutions with the same properties to [2] for semilinear equations whose principal parts are strictly hyperbolic.

§1. **Notations, definitions and a result.** Let Z_+ denote the set of all nonnegative integers and D^α stands for $(\partial/\partial z_1)^{\alpha_1}\cdots(\partial/\partial z_n)^{\alpha_n}$ where $\alpha=(\alpha_1,\cdots,\alpha_n)\in(Z_+)^n$. Moreover, let Ω be a domain of C^n containing the origin and we denote the set of all holomorphic functions on Ω by $\mathcal{O}(\Omega)$.

Now we consider the nonlinear partial differential operator $P(u)$ of differential order m and multiple order $p\geqq 2$, which we denote

$$P(u)=\sum_{\mu\in\mathcal{L}} a_\mu(z)((D^\alpha u))^\mu, \tag{1.1}$$

where $\mathcal{L}$ is a given subset of

$$\{\mu=(\mu_\alpha)\in(Z_+)^N\,;\,|\mu|=\sum_{|\alpha|\leqq m}\mu_\alpha\leqq p\}$$

with

$$N=\sharp\{\alpha\in(Z_+)^n\,;\,|\alpha|\leqq m\},\qquad a_\mu(z)\in\mathcal{O}(\Omega)\backslash\{0\}$$

and

$$((D^\alpha u))^\mu=\prod_{|\alpha|\leqq m}(D^\alpha u)^{\mu_\alpha}.$$

Then we may assume without loss of generality that $|\mu|\geqq 1$ for any $\mu\in\mathcal{L}$.

For each $\mu\in\mathcal{L}$ we put

$$y_\mu(\sigma)=|\mu|\sigma-\Sigma_\alpha|\alpha|\mu_\alpha \qquad \text{for } \sigma\in C.$$

Let $\mu\sim\nu\Leftrightarrow y_\mu=y_\nu$ and for a given $\tau\in\mathcal{L}/\sim$ denote y_τ or $|\tau|$ the common y_μ or $|\mu|$ for any $\mu\in\tau$, respectively. Let $\eta(\rho)=\min\{y_\mu(\rho)\,;\,\mu\in\mathcal{L}\}$ for $\rho\in R$. Then the graph of $\eta(\rho)$ describes a concave polygon which has a finite number, say R, of summits. We arrange the values of ρ corresponding to these summits in the order $\sigma_0<\sigma_1<\cdots<\sigma_{R-1}$ and call them characteristic exponents of the operator P.

Let $C^+=\{\omega=\exp(i\theta)\,;\,-\pi/2<\theta<\pi/2\}\subset C$.

Definition 1. (1) For a given $\sigma\in C$ and a given $\omega\in C^+$, a class $\pi\in\mathcal{L}/\sim$ is called the *principal class* of P for (σ,ω) if and only if

$$\operatorname{Re}(\omega y_\pi(\sigma))<\operatorname{Re}(\omega y_\nu(\sigma)) \qquad \text{for any } \nu\in\mathcal{L}\backslash\pi.$$

(2) For a characteristic exponent σ_r, $0\leqq r\leqq R-1$, the set $\{\mu\in\mathcal{L}\,;\,y_\mu(\sigma_r)=\eta(\sigma_r)\}$ is called the *principal class* for the characteristic exponent σ_r and denoted by π_{σ_r}.

For any $z\in C$ and any $k\in Z$, we denote $[z:k]=z(z-1)\cdots(z-k+1)$ for $k>0$, $[z:0]=1$ and $[z:k]=0$ otherwise.

Definition 2. (1) Let π be a principal class for some $(\sigma,\omega)\in C\times C^+$. The polynomial

$$p_\pi(s,z,\xi)=\textstyle\sum_{\mu\in\pi}a_\mu(z)(([s:|\alpha|]\xi^\alpha))^\mu$$

of $s\in C$ and $\xi\in C^n$, is called the *characteristic polynomial* for the class π of the operator P.

(2) For every characteristic exponent σ_r, $0\leqq r\leqq R-1$, the *characteristic polynomial* for the characteristic exponent σ_r of P is defined by the polynomial

$$p_{\sigma_r}(z,\xi,\chi)=\textstyle\sum_{\mu\in\pi_{\sigma_r}}a_\mu(z)(([\sigma_r:|\alpha|]\xi^\alpha))^\mu\chi^{|\mu|}$$

of $\xi\in C^n$ and $\chi\in C$.

Let

$$S=\{\phi(z)=0\} \tag{1.2}$$

be a regular hypersurface in Ω through the origin with an irreducible defining function $\phi(z)\in\mathcal{O}(\Omega)$. For a given $\omega\in C^+$ we design a $\psi(z)\in\mathcal{O}(R(\Omega\backslash S))$ the set of all holomorphic functions on the covering space $R(\Omega\backslash S)$ of $\Omega\backslash S$, by the relation

$$\phi(z)=(\psi(z))^\omega. \tag{1.3}$$

Definition 3. For a $z'\in S$, we say a sequence $\{z\}$ in $\Omega\backslash S$ is *spirally convergent* of exponent ω to z' if and only if $\{z\}$ tends to z' with the constraint $|\arg\psi(z)|<K$ for some $K>0$. We denote this simply by $z\xrightarrow[(\omega,K)]{}z'$. Moreover, we denote $(\omega,K)\text{-}\lim_{z\to z'}f(z)=A$ if and only if $f(z)$ converges A for any sequence $\{z\}$ such that $z\xrightarrow[(\omega,K)]{}z'$.

Definition 4. Let $\omega\in C^+$, $\sigma(z)\in\mathcal{O}(\Omega)$ and $\sigma(z)\not\equiv 0$. Then we say that $u(z)$ has *regular singularities of exponent $\sigma(z)$ with spiral exponent ω* on S if and only if $u(z)$ has the following form.

$$u(z)=(\phi(z))^{\sigma(z)}(F_0(z)+F_1(z)), \tag{1.4}$$

where $F_0(z)\in\mathcal{O}(\Omega)$, $F_0(0)\neq 0$, and $F_1(z)$ is holomorphic on $R(\Omega\backslash S)\cap\{|\arg\psi|$

$<K\}\cap\{0<|\psi|<\delta\}$ for any $K>0$ and some $\delta=\delta(K)>0$ satisfying (ω, K)-$\lim_{z\to z'} F_1(z)=0$ for any $z'\in S$.

Now we consider a solution $u(z)$ of the nonlinear partial differential equation

(1.5) $$P(u)=0,$$

which has regular singularities on S for some $\sigma(z)$ and ω. Then, what condition should be necessary on S?

Theorem. *Suppose that the nonlinear partial differential equation* (1.5) *admits a solution* (1.4) *with regular singularities of exponent $\sigma(z)$ with spiral exponent ω on $S=\{\phi(z)=0\}$ for an appropriate $\phi(z), \sigma(z), F_0(z), F_1(z)$ and ω.*

Then we have the following statements.

(i) *If $\sigma(z)\not\equiv\sigma_r$ for any $0\leqq r\leqq R-1$ and if π is the principal class of P for $(\sigma(0), \omega)$, then we have*

$$p_\pi(\sigma(z'), z', D\phi(z'))=0 \qquad \text{on } S.$$

(ii) *If $\sigma(z)\equiv\sigma_r$ is a characteristic exponent, then we have*

$$p_{\sigma_r}(z', D\phi(z'), F_0(z'))=0 \qquad \text{on } S.$$

Remark 1. The above theorem shows us that if $\sigma(z)\not\equiv\sigma_r$ and if $p_\pi(\sigma(z), z, \xi)\not\equiv 0$, the surface $S=\{\phi(z)=0\}$, which carries the designated singularities of the solution, must be an integral surface of the first order partial differential equation $p_\pi(\sigma(z), z, D\phi(z))=0$. Since then $p_\pi(\sigma(z), z, \xi)$ is homogeneous on ξ, we can see any bicharacteristic curve issuing from S keeps staying on S. This phenomenon is similar to that of the linear case but for the dependency of S on $\sigma(z)$. On the other hand, if $\sigma(z)$ equals identically a characteristic exponent σ_r, the surface S, which is an integral surface of the inhomogeneous equation $p_{\sigma_r}(z, D\phi(z), F_0(z))=0$, intersects to any bicharacteristic curves transversally.

Remark 2. Since we can prescribe the surface a priori, this is the case which we may be able to construct a solution of (1.5) satisfying (1.4). This problem will be treated in [3], affirmatively.

§2. Outline of the proof of the theorem. We show, first, that the characteristic polynomial $p_\pi(\sigma, z, \xi)$ or $p_{\sigma_r}(z, \xi, \chi)$ is invariant under any biholomorphic coordinate transformation. This fact admits us to take $\phi(z)=z_1$ without loss of generality. Next, we investigate the spiral limit

(2.1) $$(\omega, K-\varepsilon)\text{-}\lim_{z\to z'} z_1^{-\sigma(z)+|\alpha|}D^\alpha(z_1^{\sigma(z)}F(z))$$

for any $z'\in S$, $K>0$ and sufficiently small $\varepsilon>0$, where $F(z)=F_0(z)+F_1(z)$ satisfies the conditions of (1.4). Take a sufficiently small multi-circle centered at z, $z_1\neq 0$, whose radius about z_1 equals to $\varepsilon|z_1|$ and represent $F(z)$ in the form of iterated integral on this multi-circle. Then we can see by absolute estimates the value of (2.1) equals to

$$\delta_{|\alpha|,\alpha_1}[\sigma(z'):|\alpha|]F_0(z'),$$

where $\delta_{m,n}$ is the Kronecker's delta. Further, we have

$$(\omega, K)\text{-}\lim_{z\to z'} z_1^{y_\pi(\sigma(z))-y_\mu(\sigma(z))}=1 \quad \text{or} \quad 0$$

for $\mu\in\pi$ or $\mu\in\mathcal{L}\backslash\pi$, respectively. Then we can show the following lemma.

Lemma. *Let* $u(z)=(\phi(z))^{\sigma(z)}(F_0(z)+F_1(z))$ *be a function which has a regular singularities of exponent* $\sigma(z)$ *with spiral exponent* ω *on* $S=\{\phi(z)=0\}$. *Then we have*

(1) *If* $\sigma(z)\not\equiv\sigma_r$ *for any* r, $0\leqq r\leqq R-1$, *and if* π *is the principal class for* $(\sigma(0),\omega)$, *it holds that*

$$(\omega, K)\text{-}\lim_{z\to z'}(\phi(z))^{-y_\pi(\sigma(z))}P(u(z))=p_\pi(\sigma(z'), z', D\phi(z'))(F_0(z))^{|\pi|}$$

for any $z'\in S$ *and* $K>0$.

(2) *If* $\sigma\equiv\sigma_r$ *for some* r, *we have for any* $z'\in S$ *and* $K>0$

$$(0, K)\text{-}\lim_{z\to z'}(\phi(z))^{-\eta(\sigma_r)}P(u(z))=p_{\sigma_r}(z', D\phi(z'), F_0(z')).$$

The theorem can be proved easily from the above lemma and this completes the proof.

References

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