21. *Local Cohomology and the Absence of Poincaré Lemma in Tangential Cauchy-Riemann Complexes*

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We study the cohomology groups of the tangential Cauchy-Riemann complex with coefficients in microfunctions. In the section 1 of this note we give a sufficient condition for the non-vanishing of local cohomology groups with supports in certain closed subset. In the section 2 we show that, under some geometric condition, the Poincaré lemma fails for tangential Cauchy-Riemann complex with coefficients in microfunctions.

1. Local cohomology. Let X be a complex manifold of dimension n. Let $\mathcal{O}_X$ be the sheaf on X of holomorphic functions. Let Ω be an open subset of X and F the closed subset $X-\Omega$. We denote by $\mathcal{H}^k_F(\mathcal{O}_X)$, for $k=1,2,\cdots,n$, the local cohomology sheaves on X with supports in F. Let P be a boundary point of Ω.

Theorem 1. *Assume that there exists a germ of complex subvariety V of codimension q passing through the point P which satisfies the following conditions*

(i) *V is a complete intersection in U*

(ii) $(V\cap U)\cap\Omega=\phi$

for some neighborhood U of P.

Then at least one of the local cohomology groups $\mathcal{H}^1_F(\mathcal{O}_X)$, $\mathcal{H}^2_F(\mathcal{O}_X)$, $\cdots$, $\mathcal{H}^q_F(\mathcal{O}_X)$ does not vanish at P.

The proof is based on the fundamental properties of the generalized Bochner-Martinelli form (cf. [10]).

The following corollary is a natural generalization of a result of Andreotti-Norguet [1].

Corollary 2. *If, under the assumptions of Theorem 1, $\mathcal{H}^1_F(\mathcal{O}_X)_P=\mathcal{H}^2_F(\mathcal{O}_X)_P=\cdots=\mathcal{H}^{q-1}_F(\mathcal{O}_X)_P=0$, then $\mathcal{H}^q_F(\mathcal{O}_X)_P\neq 0$.*

Regarding $\{P\}$ as a complex submanifold of codimension n, we have the following

Corollary 3. *Let P be a boundary point of Ω. If $\mathcal{H}^2_F(\mathcal{O}_X)_P=\cdots=\mathcal{H}^n_F(\mathcal{O}_X)_P=0$, then $\mathcal{H}^1_F(\mathcal{O}_X)_P\neq 0$.*

Note that Corollary 3 is a local cohomological version of a result of Hörmander Theorem (Th. 4.2.9 of [5]).

2. Tangential Cauchy-Riemann complex. Let $\Omega=\{z\mid\rho(z,\bar{z})<0\}$ be a domain in X with real analytic boundary N. Here ρ is a real-valued real analytic function. (We assume that the gradient grad ρ of ρ does not vanish on N.) F denotes the closed subset $\{z\mid\rho(z,\bar{z})\geqq 0\}$. Let Y be a complexification of N and let S_N^*Y be the spherical conormal bundle.

We regard $S_N^*X = N_+ \coprod N_-$ as a subset of S_N^*Y, where N_+ (resp. N_-) is the set of unit exterior (resp. interior) conormal vectors to N. The point of N_+ which is the unit exterior conormal vector to N at $P \in N$ will be denoted by P_+. Let $\bar{\partial}_b$ be the tangential Cauchy-Riemann system induced on N.

As an application of Theorem 1, we have the following result.

Theorem 4. *Assume that there exists a germ of complex subvariety V of codimension q passing through the point $P \in N$ which satisfies the following conditions:*

(i) *V is a complete intersection in U*

(ii) $(V \cap U) \cap \Omega = \phi$

for some neighborhood U of P.

Then at least one of the cohomology groups

$$\mathcal{H}om_{\mathcal{E}_Y}(\bar{\partial}_b, C_N), \qquad \mathcal{E}xt^1_{\mathcal{E}_Y}(\bar{\partial}_b, C_N), \cdots, \mathcal{E}xt^{q-1}_{\mathcal{E}_Y}(\bar{\partial}_b, C_N)$$

does not vanish at P_+. Here C_N is the sheaf on S_N^*Y of microfunctions and $\mathcal{E}_Y$ is the sheaf of rings of pseudo-differential operators.

We also have the following

Corollary 5 (cf. Catlin [3], Diederich-Pflug [4]). *If, under the assumptions of Theorem 4,*

$$\mathcal{H}om_{\mathcal{E}_Y}(\bar{\partial}_b, C_N)_{P_+} = \mathcal{E}xt^1_{\mathcal{E}_Y}(\bar{\partial}_b, C_N)_{P_+} = \cdots = \mathcal{E}xt^{q-2}_{\mathcal{E}_Y}(\bar{\partial}_b, C_N)_{P_+} = 0,$$

we have $\mathcal{E}xt^{q-1}_{\mathcal{E}_Y}(\bar{\partial}_b, C_N)_{P_+} \neq 0$.

Example 6. Let $X = C^3$ and let $\rho = (1/2)(z_1 + \bar{z}_1) + |z_2|^2 + |z_1|^2 |z_3|^2$. Set $\Omega = \{(z_1, z_2, z_3) \in X \mid \rho > 0\}$, $N = \{(z_1, z_2, z_3) \mid \rho = 0\}$. Let $P = (0, 0, 0)$. Since the complex submanifold $V = \{(z_1, z_2, z_3) \mid z_1 = z_2 = 0\}$ is contained in the closed set F, we have

$$\mathcal{H}om_{\mathcal{E}_Y}(\bar{\partial}_b, C_N)|_{N_+} = 0, \qquad \mathcal{E}xt^1_{\mathcal{E}_Y}(\bar{\partial}_b, C_N)_{P_+} \neq 0.$$

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