16. *Class Number One Problem for Real Quadratic Fields*

(The conjecture of Gauss)

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The following conjecture of Gauss on the class number of real quadratic fields is well known:

(G_1): There exist infinitely many real quadratic fields of class number one, or more precisely

(G_2): There exist infinitely many real quadratic fields $Q(\sqrt{p})$ of class number one such that p is prime congruent to 1 mod 4.

In relation to this conjecture of Gauss, the following conjecture of S. Chowla and analogous conjecture of Yokoi are known[1]:

(C_1) (S. Chowla): Let D be a square-free rational integer of the form $D=4m^2+1$ for natural number m. Then, there exist exactly 6 real quadratic fields $Q(\sqrt{D})$ of class number one,

i.e. $(D, m)=(5, 1), (17, 2), (37, 3), (101, 5), (197, 7), (677, 13)$.

(C_2) (H. Yokoi): Let D be a square-free rational integer of the form $D=m^2+4$ for natural number m. Then, there exist exactly 6 real quadratic fields $Q(\sqrt{D})$ of class number one,

i.e. $(D, m)=(5,1), (13,3), (29,5), (53,7), (173,13), (293,17)$.

Concerning the conjectures (C_1), (C_2), R. A. Mollin says[2]: Conjecture (C_1) was proved under the assumption of the generalized Riemann hypothesis in [6], and conjecture (C_2) also can be proved under the same assumption in a similar way.

On the other hand, H. K. Kim, M. G. Leu and T. Ono[3] recently proved that at least one of the two conjectures (C_1), (C_2) is true and that for the other case there are at most 7 quadratic fields $Q(\sqrt{D})$ of class number one by using results of Tatuzawa [1], Yokoi [3] and by the help of a computer.

Let $\varepsilon_D=(1/2)(t_D+u_D\sqrt{D})>1$ be the fundamental unit of the real quadratic field $Q(\sqrt{D})$ for a positive square-free integer D. Then, (C_1) is a conjecture on real quadratic fields $Q(\sqrt{D})$ with $u_D=2$, and (C_2) is a conjecture on real quadratic fields $Q(\sqrt{D})$ with $u_D=1$.

In this paper, we shall prove first the following theorem on real

1) cf. S. Chowla and J. Friedlander [2] and H. Yokoi [3].

2) cf. R. A. Mollin [4].

3) cf. H. K. Kim, M. G. Leu and T. Ono [5].

quadratic fields $Q(\sqrt{D})$ with general u_D in the case of prime D congruent to 1 mod 4:

Theorem 1. *Put*

$$U=\{2^{\delta}\prod_i p_i^{e_i};\ \delta=0 \text{ or } 1,\ e_i\geqq 1,\ \text{prime } p_i\equiv 1\ (4)\}.$$

Then, for any fixed u in U, there exists only a finite number of real quadratic fields $Q(\sqrt{p})$ of class number one such that p is prime congruent to 1 mod 4 and $u_p=u$ for the fundamental unit $\varepsilon_p=(1/2)(t_p+u_p\sqrt{p})>1$ of $Q(\sqrt{p})$.

To prove this theorem, we need two lemmas.

Lemma 1 (Tatuzawa)[4]. *For any positive number c satisfying $1/2>c>0$, let d be any positive integer such that $d\geqq\max(e^{1/c}, e^{11.2})$. Moreover, let χ be any non-principal primitive real character to modulus d, and $L(s,\chi)$ be the corresponding L-series.*

Then, $L(1,\chi)>0.655(c/d^c)$ holds with one possible exception.

Lemma 2. *Let $\varepsilon_d=(1/2)(t+u\sqrt{d})>1$ be the fundamental unit of a real quadratic field $Q(\sqrt{d})$. Then, in the case $N\varepsilon_d=+1$, it holds $t>\varepsilon_d>u\sqrt{d}$, and in the case $N\varepsilon_d=-1$, it holds $t<\varepsilon_d<u\sqrt{d}$.*

Proof. Since $N\varepsilon_d=\pm 1$ implies $t^2-du^2=\pm 4$, in the case $N\varepsilon_d=1$, we get at once

$$t>\varepsilon_d=\frac{1}{2}(t+u\sqrt{d})>u\sqrt{d}$$

from $t=\sqrt{du^2+4}>u\sqrt{d}$. Similarly, in the case $N\varepsilon_d=-1$, we get

$$t<\varepsilon_d=\frac{1}{2}(t+u\sqrt{d})<u\sqrt{d}$$

from $t=\sqrt{du^2-4}<u\sqrt{d}$.

Proof of Theorem. If we put $c=1/m$ for any m satisfying $m\geqq 11.2$, then $\max(e^{1/c}, e^{11.2})=e^m$ holds, and hence it follows from Lemma 1 that

$$L(1,\chi_p)>\frac{0.655}{m}p^{-1/m},$$

where χ_p is the Kronecker character belonging to the quadratic field $Q(\sqrt{p})$ and $L(s,\chi_p)$ is the corresponding L-series.

On the other hand, since $N\varepsilon_p=-1$ holds for prime $p\equiv 1 \pmod 4$, it follows from Dirichlet's class number formula and Lemma 2 that for the class number $h(p)$ of $Q(\sqrt{p})$

$$\begin{aligned} h(p)&=\frac{\sqrt{p}}{2\log\varepsilon_p}L(1,\chi_p)\\ &>\frac{\sqrt{p}}{2\log u_p\sqrt{p}}\,\frac{0.655}{m}p^{-1/m}\\ &=\frac{0.655}{m}\,\frac{1}{2\log u_p+\log p}p^{(m-2)/2m}. \end{aligned}$$

Here, if we put

4) See Tatuzawa [1] for proofs.

$$f(x)=\frac{x^{(m-2)/2m}}{2\log u+\log x},$$

then $(m-2)/2m>0$ implies $\lim_{x\to\infty} f(x)=\infty$.

Hence, there exists only a finite number of prime p congruent to 1 mod 4 such that $u_p=u$ and $h(p)=1$ hold. Thus our proof of theorem 1 was completed.

As an application of this theorem, we can prove easily the following theorem:

Theorem 2. *The conjecture (G_2) of Gauss is equivalent to the following conjecture:*

(G_3): *For any given natural number u_0, there exists at least one real quadratic field $Q(\sqrt{p})$ of class number one such that p is prime congruent to* 1 mod 4 *and $u_p\geqq u_0$ for the fundamental unit $\varepsilon_p=(1/2)(t_p+u_p\sqrt{p})>1$ of $Q(\sqrt{p})$.*

Proof. From $-4=4N\varepsilon_p=t_p^2-pu_p^2$, we get easily that for any odd prime factor q of u_p it holds $(-1/q)=1$, where $(-1/q)$ means Legendre-Jacobi symbol. Hence, since $q\equiv 1 \pmod 4$, we know $u_p\in U$. Therefore, by Theorem 1, it is clear that (G_2) implies (G_3).

On the other hand, since it is trivial that (G_3) implies (G_2), our proof of theorem 2 was completed.

References

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