2. Radial Non-positive Solutions for Nonlinear Equation $\Delta u+|x|^l|u|^{p-1}u=0$ on the Ball

By Ken'ichi NAGASAKI
Department of Mathematics, Chiba Institute of Technology

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§ 1. Introduction and results. In this paper we study the radial solutions of the nonlinear boundary value problem

$$\text{(P)}\qquad \begin{cases} \Delta u+|x|^l|u|^{p-1}u=0 & \text{in } \Omega=\{x\,|\,|x|<1\}\subset \boldsymbol{R}^n, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where $n\geqq 3$, $l\geqq 0$ and $p>1$.

The radial solution $u=u(r)$ of (P), where $r=|x|$, can be obtained as a solution of the following ordinary differential equation

$$(1)\qquad \begin{cases} (r^{n-1}u')'+r^{l+n-1}|u|^{p-1}u=0, & r\in(0,1), \\ u'(0)=0,\ u(1)=0. \end{cases}$$

By refining the phase plane method developed in [1] and [4], we can show the following

Theorem. *If $p\in(1,(n+2+2l)/(n-2))$, for each positive integer k there exists a unique radial solution $u=u(r)$ of* (P), *such that $u(0)>0$ and $u(r)$ has exactly $k-1$ zeros in* $(0,1)$.

If $p\in[(n+2+2l)/(n-2),+\infty)$, there exists no radial solution of (P) *except the trivial one.*

For $l=0$ and $p>1$, Gidas-Ni-Nirenberg [3] has shown the uniqueness of the positive radial solution of (P). When the domain is an annulus $\{x\,|\,a<|x|<b\}\subset \boldsymbol{R}^n$, $l=0$ and $p\in(1,(n+2)/(n-2)]$, the unique existence of a positive radial solution of (P) is established in Ni [6]. Further, a similar result to our theorem is obtained by Ni-Nussbaum [7] when Ω is an annulus, $l\in\boldsymbol{R}$, $p>1$ and $n\geqq 2$. Also for our problem, Ni [5] showed the existence of a positive radial solution of (P) for $p\in(1,(n+2+2l)/(n-2))$ applying the Mountain Pass Lemma, but did not get the uniqueness.

§ 2. Outline of the proof of Theorem. For the moment, we consider the initial value problem

$$(2)\qquad (r^{n-1}u')'+r^{l+n-1}|u|^{p-1}u=0, \qquad r\in(0,1),$$

$$(3)\qquad u(0)=A, \qquad u'(0)=0,$$

for $A>0$, instead of the boundary value problem (1).

Following after Chandrasekhar [1], we introduce the change of variables

$$(4)\qquad u(r)=Av(s), \qquad r=Bs,$$

where a constant B is determined as follows. In the case I $p\in((n+l)/(n-2),+\infty)$, $B=\{(n-2-\tau)\tau A^{1-p}\}^{1/(2+l)}$ with $\tau=(2+l)/(p-1)$, in the case II $p=(n+l)/(n-2)$, $B=(A^{1-p})^{1/(2+l)}$ and in the case III $p\in(1,(n+l)/(n-2))$,

$B=\{(\tau+2-n)\tau A^{1-p}\}^{1/(2+l)}$.

According to the cases I, II and III, the equation (2) is transformed respectively into the equations

$$(5)_{\mathrm{I}} \qquad (s^{n-1}v')'+(n-\tau-2)\tau s^{l+n-1}|v|^{p-1}v=0,$$

$$(5)_{\mathrm{II}} \qquad (s^{n-1}v')'+s^{l+n-1}|v|^{p-1}v=0$$

and

$$(5)_{\mathrm{III}} \qquad (s^{n-1}v')'+(\tau+2-n)\tau s^{l+n-1}|v|^{p-1}v=0,$$

while the initial condition (3) is transformed into

$$(6) \qquad v(0)=1, \qquad v'(0)=0.$$

Here, the boundary condition $u(1)=0$ corresponds to the condition $v(B^{-1})=0$.

Next we make the change of variables such as

$$(7) \qquad w(t)=s^{\tau}v(s), \qquad s=e^{t},$$

which transforms the equations $(5)_{\mathrm{I}}$, $(5)_{\mathrm{II}}$ and $(5)_{\mathrm{III}}$ respectively into the equations

$$(8)_{\mathrm{I}} \qquad w''+(n-2\tau-2)w'+(n-\tau-2)\tau(|w|^{p-1}-1)w=0,$$

$$(8)_{\mathrm{II}} \qquad w''-(n-2)w'+|w|^{p-1}w=0$$

and

$$(8)_{\mathrm{III}} \qquad w''+(n-2\tau-2)w'+(\tau+2-n)\tau(|w|^{p-1}+1)w=0.$$

The initial condition (6) is transformed into the condition

$$(9) \qquad \lim_{t\to-\infty} e^{-\tau t}w(t)=1, \qquad \lim_{t\to-\infty} e^{-t}\{e^{-\tau t}w(t)\}'=0.$$

The unique existence of the solution satisfying (8) and (9) follows immediately from

Lemma 1. *Let constants α and β satisfy $\alpha>0$ and $\alpha>\beta$. Assume f: $\boldsymbol{R}\to\boldsymbol{R}$ to be C^1 and satisfy for some $\varepsilon>\alpha^{-1}$,*

$$f(t)=0(t^{1+\varepsilon}), \quad f'(t)=0(t^{\varepsilon}) \quad \text{as } t\to 0.$$

Then we have a positive T such that the problem

$$\varphi''-(\alpha+\beta)\varphi'+\alpha\beta\varphi+f(\varphi)=0,$$

$$\lim_{t\to-\infty} e^{-\alpha t}\varphi(t)=1, \qquad \lim_{t\to-\infty} e^{-t}\{e^{-\alpha t}\varphi(t)\}'=0,$$

has a unique solution $\varphi=\varphi(t)$ in $(-\infty, -T)$.

For the solution $w=w(t)$ of (8) and (9), we put $z(t)=w'(t)$ and trace the orbit $\mathcal{O}=\{(w(t), z(t))\,|\,-\infty<t<+\infty\}$ in (w, z)-phase plane.

First we deal with the case I where $p\in((n+l)/(n-2), +\infty)$. In this case, the orbit $\mathcal{O}$ tends to the origin along the line $z=\tau w$ from positive w as $t\to-\infty$. Further, there is no orbit having such a property other than $\mathcal{O}$ ([2] chapter 15, Theorem 6.1.). From this we know

Proposition 1. (i) *For $p\in((n+2+2l)/(n-2), +\infty)$, the orbit $\mathcal{O}$ never meets the z-axis. It approaches to* (1, 0) *as $t\to+\infty$.*

(ii) *For $p=(n+2+2l)/(n-2)$, $\mathcal{O}$ forms a ring which starts from the origin along $z=\tau w (w>0)$ and terminates at the origin along $z=-\tau w (w>0)$.*

(iii) *For $p\in((n+l)/(n-2), (n+2+2l)/(n-2))$, $\mathcal{O}$ goes away from the origin crossing the w-axis and z-axis alternately.*

Here we note that for $p=(n+2+2l)/(n-2)$ the solution $u=u(r)$ of (2) and (3) can be explicitly expressed as $u(r)=A(1+Cr^{l+2})^{-(n-2)/(l+2)}$ where $C=(1/(n-2)(n+l))A^{(4+2l)/(n-2)}$.

Similarly, we also have the following

Proposition 2. *For* $p\in(1,(n+l)/(n-2)]$, *the orbit* $\mathcal{O}$ *behaves the same as in Proposition* 1-(iii).

Detailed proof of Lemma and Propositions will be published elsewhere.

Now we are going to see how the trace of the orbit $\mathcal{O}$ tells us about the radial solution of (P). In fact, in terms of the changes of variables (4) and (7), the zeros of u correspond to that of w. Hence, if $\mathcal{O}$ never meets the z-axis, the solution $u=u(r)$ of (2) and (3) never vanishes. On the other hand, if $\mathcal{O}$ goes across the z-axis at $t=\bar{t}$, the solution $u=u(r)$ for A determined by the relation $B^{-1}=e^{\bar{t}}$ in each case vanishes at $r=1$. Moreover, if $\mathcal{O}$ meets the z-axis $k-1$ times for $t\in(-\infty,\bar{t})$, the above solution $u=u(r)$ has just $k-1$ zeros in (0, 1). At this point Theorem can be proved easily.

§3. **Remark.** From Rellich's identity we can show that the following identity is valid for any solution of (P).

$$\frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial n}\right|^2 dx=\left(\frac{n+l}{p+1}-\frac{n-2}{2}\right)\int_{\Omega}|x|^l|u|^{p+1}dx,$$

where n denotes the outward unit normal on $\partial\Omega$. Using this identity, we have shown the stronger result than the second part of Theorem, that is, for $p\in[(n+2+2l)/(n-2),+\infty)$, the problem (P) has only the trivial solution.

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