13. On Pathwise Projective Invariance of Brownian Motion. I[†), *)]

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Introduction. Brownian motion with parameter in Riemannian space was introduced by P. Lévy [3]. He also considered white noise representation of Brownian motion in connection with geometric structure of its parameter space. In line with his idea we start with the simplest case of usual 1-parameter Brownian motion. The parameter space is considered the projective space $\boldsymbol{P}^1$ rather than $\boldsymbol{R}^1$.

In part I, we study an invariance property of the path space. This property is a reflection of the projective structure of $\boldsymbol{P}^1$. We also see that this invariance characterizes the Brownian motion between 1-parameter self-similar Gaussian processes.

In part II, the type of the group action which describes the above invariance will be determined as a *discrete series representation of index 2* in term of the theory of unitary representation.

In part III, we will consider a generalization of the partially invariance in § 3. Proposition 4 will be extended to multi-parameter case. The Möbius group will appear in the invariance property.

§ 1. Projective invariance. A Gaussian system $\{B(t;\omega);t\in\boldsymbol{R}\}$ is called a Brownian motion if it satisfies

($\mathscr{B}1$) $B(0)\equiv 0$,

($\mathscr{B}2$) $B(t)-B(s)\overset{\mathcal{L}}{=}N(0,|t-s|)$, the Gaussian law of mean 0 and variance $|t-s|$.

To fix the idea, take a continuous version

($\mathscr{B}3$) $B(t;\omega)$ is continuous in t including $t=\infty$ for any ω, that is
$$\lim_{|t|\to\infty}\frac{1}{t}B(t)=0.$$

It is easy to show that the processes $B_{1,s}(t)$, $B_{2,u}(t)$ and $B_3(t)$ below are Brownian motions in the above sense;

($\mathscr{T}1$) $B_{1,s}(t)\equiv B(t+s)-B(s)$, $s\in\boldsymbol{R}$,

($\mathscr{T}2$) $B_{2,u}(t)\equiv e^{-u/2}B(e^u t)$, $u\in\boldsymbol{R}$,

($\mathscr{T}3$) $B_3(t)\equiv tB\left(\dfrac{-1}{t}\right)$.

It is natural to ask what group is generated by the transforms ($\mathscr{T}1$)–($\mathscr{T}3$) acting on $B(t;\omega)$.

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Theorem 1. (i) *For any* $g=\begin{pmatrix}a, b\\ c, d\end{pmatrix}\in SL(2, \boldsymbol{R})$, *the process*

$$B^{g}(t;\omega)\equiv(ct+d)B\left(\frac{at+b}{ct+d};\omega\right)-ctB\left(\frac{a}{c};\omega\right)-dB\left(\frac{b}{d};\omega\right) \tag{1}$$

is a Brownian motion.

(ii) $(B^{g})^{h}(t;\omega)\equiv B^{gh}(t;\omega)$ *holds for any* $g, h\in SL(2,\boldsymbol{R})$ *and almost all* ω.

Proof. The group $SL(2;\boldsymbol{R})$ is locally isomorphic to the group generated by ($\mathcal{T}$1)-($\mathcal{T}$3). The essential part of the proof is to check of the iteration law (ii). We can check it by direct calculations. For example, let $g=\begin{pmatrix}a, b\\ c, d\end{pmatrix}$ and $J=\begin{pmatrix}0, & -1\\ 1, & 0\end{pmatrix}$.

$$(B^{g})^{J}(t)=t\left\{\left(-\frac{c}{t}+d\right)B\left(\frac{-a/t+b}{-c/t+d}\right)+\frac{c}{t}B\left(\frac{a}{c}\right)-dB\left(\frac{b}{d}\right)\right\}$$

$$=(dt-c)B\left(\frac{bt-a}{dt-c}\right)-dtB\left(\frac{b}{d}\right)-(-c)B\left(\frac{-a}{-c}\right)=B^{gJ}(t).$$

The continuity condition ($\mathcal{B}$3) is easily checked.

§2. Lévy's projective invariance. Let $[\alpha, \beta]$ be an interval and $t\in(\alpha,\beta)$. Lévy's normalized Brownian bridge $\xi^{[\alpha,\beta]}(t)$ is defined;

$$\xi^{[\alpha,\beta]}(t)\equiv\mathcal{N}\{B(t)-B(\alpha)-\boldsymbol{E}[B(t)-B(\alpha)\,|\,B(\beta)-B(\alpha)]\} \tag{2}$$

$$=\sqrt{\frac{\beta-\alpha}{(t-\alpha)(\beta-t)}}B(t)-\sqrt{\frac{\beta-t}{(\beta-\alpha)(t-\alpha)}}B(\alpha)-\sqrt{\frac{t-\alpha}{(\beta-\alpha)(\beta-t)}}B(\beta),$$

where $\mathcal{N}$ is the normalizing constant which makes $\xi^{[\alpha,\beta]}(t)$ a standard Gaussian random variable.

Let $g=\begin{pmatrix}a, b\\ c, d\end{pmatrix}^{-1}\in SL(2,\boldsymbol{R})$, and $[\tilde{\alpha},\tilde{\beta}]$ be the image of $[\alpha,\beta]$. That is,

$$\alpha=\frac{a\tilde{\alpha}+d}{c\tilde{\alpha}+d}\quad\text{and}\quad\beta=\frac{a\tilde{\beta}+d}{c\tilde{\beta}+d}.$$

Let above normalization be applied to the process $B^{g}(t)$.

$$\xi^{g,[\tilde{\alpha},\tilde{\beta}]}(t)$$

$$=\sqrt{\frac{\tilde{\beta}-\tilde{\alpha}}{(t-\tilde{\alpha})(\tilde{\beta}-t)}}B^{g}(t)-\sqrt{\frac{\tilde{\beta}-t}{(\tilde{\beta}-\tilde{\alpha})(t-\tilde{\alpha})}}B^{g}(\tilde{\alpha})-\sqrt{\frac{t-\tilde{\alpha}}{(\tilde{\beta}-\tilde{\alpha})(\tilde{\beta}-t)}}B^{g}(\tilde{\beta}),$$

$$=\sqrt{\frac{\tilde{\beta}-\tilde{\alpha}}{(t-\tilde{\alpha})(\tilde{\beta}-t)}}\left\{(ct+d)B\left(\frac{at+b}{ct+d}\right)-ctB\left(\frac{a}{c}\right)-dB\left(\frac{b}{d}\right)\right\}-\cdots-\cdots$$

$$=\sqrt{\frac{(d\beta-b)/(-c\beta+a)-(d\alpha-b)/(-c\alpha+a)}{[\{t(-c\alpha+a)-(d\alpha+b)\}/(-c\alpha+a)][\{(d\beta+b)-t(-c\beta+a)\}/(-c\beta+a)]}}$$

$$\times(ct+d)B\left(\frac{at+b}{ct+d}\right)-\cdots-\cdots$$

$$=\sqrt{\frac{\beta-\alpha}{\{(at+b)-\alpha(ct+d)\}\{\beta(ct+d)-(at+b)\}}}\,|ct+d|\,sgn(ct+d)B(\cdot\cdot)-\cdots$$

(*set* $s=(at+b)/(ct+d)$)

$$=\sqrt{\frac{\beta-\alpha}{(s-\alpha)(\beta-s)}}\,sgn(ct+d)B(s)-\cdots-\cdots$$

$$=\varepsilon\xi^{[\alpha,\beta]}(g^{-1}t;\omega),\qquad\varepsilon=sgn(ct+d).$$

Thus, we obtain

Theorem 2. $\xi^{g,[g\alpha, g\beta]}(gs\,;\omega)=\varepsilon\xi^{[\alpha,\beta]}(s\,;\omega), \qquad \varepsilon=\pm 1.$

As a corollary, we get the projective invariance property of P. Lévy.

Corollary 3 (P. Lévy [3]).

$$E[\xi^{[\alpha,\beta]}(t)\,\xi^{[\alpha,\beta]}(s)]=E[\xi^{[g\alpha, g\beta]}(gt)\cdot\xi^{[g\alpha, g\beta]}(gs)].$$

§ 3. Partial invariance of self-similar processes. For any α, $0<\alpha<2$, there exists a Gaussian process $X^\alpha(t)$, $t\in \boldsymbol{R}$, called self-similar process of index α which satisfies the following conditions:

($\mathcal{S}1$) $X^\alpha(0)=0$,

($\mathcal{S}2$) $\boldsymbol{E}|X^\alpha(t)-X^\alpha(s)|^2=|t-s|^\alpha$.

($\mathcal{S}3$) $X^\alpha(t\,;\omega)$ is continuous in t for almost all ω.

Let us consider the following transformations of the path of X^α,

($\mathcal{T}1'$) $Y_1^\alpha(t)\equiv X^\alpha(t+s)-X^\alpha(s)$, $s\in\boldsymbol{R}$,

($\mathcal{T}2'$) $Y_2^\alpha(t)\equiv e^{-u\alpha/2}X^\alpha(e^u t)$, $u\in\boldsymbol{R}$ and

($\mathcal{T}3'$) $Y_3^\alpha(t)\equiv sgn^\varepsilon(t)|t|^\alpha X^\alpha(-1/t)$, $\varepsilon=0$ or 1.

We may expect that there exists a similar invariance property for self-similar processes as the case of Brownian motion. Contrary to our expectation, ($\mathcal{T}1'$)-($\mathcal{T}3'$) do not make a group.

Set

$$G_u=\left\{g=\begin{pmatrix} a, b \\ 0, 1/a\end{pmatrix}\in SL(2,\boldsymbol{R})\right\}$$

and

$$G_l=\left\{h=\begin{pmatrix} c, 0 \\ d, 1/c\end{pmatrix}\in SL(2,\boldsymbol{R})\right\}.$$

Define actions of g and h as follows,

$$X^{\alpha,g}(t)\equiv|a|^{-\alpha}X^\alpha(a^2t+ab)-|a|^{-\alpha}X^\alpha(ab) \tag{3}$$

and

$$X^{\alpha,h}(t)\equiv\left|dt+\frac{1}{c}\right|^\alpha X^\alpha\left(\frac{ct}{dt+1/c}\right)-|ct|^{-\alpha}X^\alpha(c). \tag{4}$$

Then it holds,

Proposition 4. i) *The processes $X^{\alpha,g}$ and $X^{\alpha,h}$ are self-similar processes of index α.*

ii) $(X^{\alpha,g})^{g'}(t)=X^{\alpha,gg'}(t)$ and $(X^{\alpha,h})^{h'}(t)=X^{\alpha,hh'}(t)$ hold for any $g, g'\in G_u$ and $h, h'\in G_l$.

iii) There exist $g, g'\in G_u$ and $h, h'\in G_l$ which satisfy $gh=h'g'$ as an element of $SL(2,\boldsymbol{R})$ and

$$(X^{\alpha,g})^h(t)\neq(X^{\alpha,h'})^{g'}(t). \tag{5}$$

Proof. The proofs of i) and ii) are simple so are omitted. For iii) it is enough to give an example. Let

$$g=\begin{pmatrix}1/2, & -\sqrt{3}/3\\ 0\;, & 2\end{pmatrix},\qquad g'=\begin{pmatrix}1, & -\sqrt{3}\\ 0, & 1\end{pmatrix},$$

$$h=\begin{pmatrix}1 & , 0\\ \sqrt{3}/6, & 1\end{pmatrix}\quad\text{and}\quad h'=\begin{pmatrix}1/3 & , 0\\ \sqrt{3}/3, & 3\end{pmatrix}.$$

It is easy to see that the above elements give us an example of (5).

Note. Even the case of Brownian motion, if we take one of the transforms $\tilde{B}_3(t) \equiv |t| B(1/t)$, $\tilde{B}_3(t) \equiv |t| B(-1/t)$ and $\tilde{B}_3(t) \equiv t B(1/t)$ instead of ($\mathcal{T}3$), we fail to find the full group action on $B(t)$.

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