112. *Holomorphic and Harmonic Maps between Complete Almost Kaehler Manifolds*

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0. Introduction. The purpose of this paper is to study harmonic maps whose domain manifolds are non-compact.

The theories of harmonic maps whose domain manifolds are compact have been investigated. Especially, the following well known stability, rigidity theorems obtained by Lichnérowicz (cf. [3]) among the theories of such harmonic maps are essential:

Stability theorem (cf. [3, Corollary (8.15)]). *A (anti-) holomorphic map $\phi: M\to N$ between two compact almost Kaehler manifolds M, N is harmonic and it minimizes the energy in its homotopy class.*

Rigidity theorem (cf. [3, Corollary (8.19)]). *If $\phi_t: M\to N$ is a smooth deformation of a (anti-) holomorphic map $\phi: M\to N$, between two compact almost Kaehler manifolds M and N, through harmonic maps, then each ϕ_t is holomorphic.*

The above two theorems are based on the following Homotopy theorem:

Homotopy theorem (cf. [3, Theorem (8.6)]). *If M and N in a map ϕ: $M\to N$ are compact almost Kaehler manifolds, then the difference $K(\cdot)=E'(\cdot)-E''(\cdot)$ between the holomorphic energy and anti-holomorphic one is a smooth homotopy invariant, i.e., if $\phi_0, \phi_1: M\to N$ are smooth homotopic, then $K(\phi_0)=K(\phi_1)$.*

On the other hand, when domain manifold M is non-compact, many investigations of harmonic maps except for the theory of harmonic functions on complete manifold obtained by M. Anderson and R. Schoen (cf. [1]) were carried through under the assumption that the domain manifold M has a finite volume or M is locally symmetric (cf. [4, 5]). The studies of harmonic maps with non-compact domain manifold except the above conditions are rarely ever seen.

The purpose of this paper is to extend the above theorems to the case that the domain manifolds are complete almost Kaehler manifolds:

Theorem A. *Let M, N be complete almost Kaehler manifolds. Then an L^2-bounded (anti-) holomorphic map $\phi: M\to N$ minimizes the energy function in its L^2-bounded homotopy.*

Theorem B. *Let M, N be complete almost Kaehler manifolds. If $\phi_t: M\to N$ is an L^2-bounded deformation of an L^2-bounded (anti-) holomorphic map $\phi: M\to N$ through harmonic maps, then each ϕ_t is (anti-) holomorphic.*

Theorem C. *Let M, N be complete almost Kaehler manifolds. Then $K(\cdot)$ is L^2-bounded homotopy invariant.*

Moreover, as a by-product of this work, an alternative simple proof of Homotopy theorem (cf. [3], Theorem (8.6)) by Lichnérowicz is obtained.

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1. Preliminaries. Throughout this paper, let M be a complete $2m$-dimensional almost Kaehler manifold which has an almost complex structure J and an almost Hermitian metric g and the Kaehler form ω^M defined by $\omega^M(X, Y)=g(JX, Y)$, where X, Y belong to the tangent bundle TM of M. Let $(N; J, h, \omega^N)$ be a $2n$-dimensional almost Kaehler manifold, where J is a complex structure and h is an almost Hermitian metric, and ω^N is the Kaehler form (cf. [3], p. 47). Let ϕ be a smooth map from M into N. We decompose the complexified differential $d^C\phi: T^CM \rightarrow T^CN$ of ϕ as follows:

$$\partial\phi: T'M \longrightarrow T'N \qquad \bar{\partial}\phi: T''M \longrightarrow T'N$$
$$\partial\bar{\phi}: T'M \longrightarrow T''N \qquad \overline{\partial\phi}: T''M \longrightarrow T''N,$$

where $T'M$ and $T'N$ are the holomorphic tangent bundles of M and N respectively, and $T''M$, $T''N$ are the anti-holomorphic tangent bundles of M and N respectively (cf. [3], p. 47).

g extended to a complex bilinear form on T^CM induces a Hermitian form g' on $T'M$, where $g'(X, Y) := g(X, \bar{Y})$ for $X, Y \in T'M$. Similarly, we can induce Hermitian forms on $T''(M)$, $T'(N)$ and $T''(N)$. Using these almost Hermitian structures of M and N, we define the partial energy densities of ϕ as the following squares of complex norms (cf. [3], p. 48):

$$e'(\phi)=|\partial\phi|^2, \qquad e''(\phi)=|\bar{\partial}\phi|^2.$$

We easily find that the energy density $e(\phi)$ of ϕ is equal to the sum of $e'(\phi)$ and $e''(\phi)$ on M. We denote

$$E'(\phi) := \int_M e'(\phi)v_g, \qquad E''(\phi) := \int_M e''(\phi)v_g,$$

when these are defined. Here v_g is the volume element. Then $E(\phi)=E'(\phi) - E''(\phi)$, where $E(\phi)$ is the energy functional of ϕ.

Obviously, ϕ is *holomorphic* iff $E''(\phi)=0$ and *anti-holomorphic* iff $E'(\phi)=0$. We shall call $\pm$ *holomorphic* a map which is holomorphic or anti-holomorphic.

We put

$$k(\phi) := e'(\phi)-e''(\phi) \quad \text{and} \quad K(\phi) := E'(\phi)-E''(\phi)$$

when defined.

Definition 1.1. Let $C_2(M, N)$ be the collection of smooth maps ϕ of M into N such that $e(\phi)$ are integrable functions on M. Two elements ϕ, $\tilde{\phi}$ in $C_2(M, N)$ are said to be *L^2-bounded homotopic* if there exists a smooth homotopy $\Phi: M\times[0, 1]\rightarrow N$ satisfying the following conditions:

(C.1) $$\Phi(x, 0)=\phi(x) \quad \text{and} \quad \Phi(x, 1)=\tilde{\phi}(x),$$

and

(C.1) $d\Phi \in L^2(M\times[0,1])$, i.e., $\sum_{i=0}^{2m} h(d\Phi(e_i), d\Phi(e_i))$

is integrable on the product manifold $M\times[0,1]$, where $\{e_i\}_{i=0}^{2m}$ is a local orthonormal fields on $M\times[0,1]$.

Then such a homotopy Φ is called an *L^2-bounded homotopy*, and each ϕ_t $(0\leqq t\leqq 1)$ is called an *L^2-bounded variation* of $\phi\in C_2(M,N)$.

Example 1.2. Let $\boldsymbol{R}^2\times[0,1]$ have the usual Riemannian metric. Let $\Phi_1: \boldsymbol{R}^2\times[0,1]\to\boldsymbol{R}^2$ be a map defined by $\Phi_1(x,y,t)=(a_1, a_2+t^2-t)$, where a_1 and a_2 are constant numbers. Then $E(\phi_1)=E(\phi_0)=0$ and $(d\Phi_1)(\partial/\partial t)=(0,2t-1)$. Hence Φ_1 is not an L^2-bounded homotopy, but Φ_1 is a smooth homotopy. We define another homotopy $\Phi_2: \boldsymbol{R}^2\times[0,1]\to\boldsymbol{R}^2$ by $\Phi_2(x,y,t)=(\lambda_r\cdot x, \lambda_r\cdot t\cdot y)$, where λ_r is a cut off function (cf. [2, 6]) defined on $\boldsymbol{R}^2$ and r is a positive constant. Then $E(\phi_0)$ and $E(\phi_1)$ are fintite and $d\Phi_2$ is a square integrable type. Hence Φ_2 is an L^2-bounded homotopy.

2. The proof of main results. We construct a family of cut off functions whose supports exhaust M (cf. [2, 6]).

Let $\mu: [0,\infty)\to\boldsymbol{R}$ be a smooth function such that $0\leqq\mu\leqq 1$, supp $\mu\subset[0,2]$ and $\mu(t)=1$ for $t\in[0,1]$. Let $d(x)$ be the geodesic distance between $x\in M$ and a fixed a point $x_0\in M$. For every $r>0$, set $\lambda_r(x)=\mu(d(x)/r)$. We denote by B_r the geodesic ball of radius r centered at x_0. It is easy to see that for every $r>0$, λ_r is Lipschitz differentiable almost everywhere on M, and has the following properties.

$$
(2.1)\qquad
\begin{aligned}
&0\leqq\lambda_r(x)\leqq 1 && \text{for every } x\in M\\
&\operatorname{supp}\lambda_r\subset B_{2r}\\
&\lambda_r(x)=1 && \text{for every } x\in B_r\\
&\lim_{r\to\infty}\lambda_r=1\\
&|d\lambda_r(x)|\leqq C_1/r \text{ almost everywhere on } M,
\end{aligned}
$$

where $C_1>0$ is a constant independent of r. We also observe that because M is complete B_r is compact for every $r>0$.

On the other hand, the following Lemma is well known (cf. [3], p. 48):

Lemma. *If $\phi: M\to N$ is a smooth map between two almost Kaehler manifolds M, N, then $k(\phi)=\langle\omega^M, \phi', (\omega^N)\rangle$, where $\langle\ ,\ \rangle$ is the inner product induced from g and ϕ' is the dual map of the differential map $d\phi$ of ϕ.*

The proofs of main results are as follows.

Alternative proof of homotopy theorem (cf. [3], Theorem (8.6)). By Lichnérowicz: Let A and B be $2m$-dimensional and $2n$-dimensional compact almost Kaehler manifolds respectively. Let $\Phi: A\times[0,1]\to B$ be a smooth map with the conditions $\Phi(\cdot,0)=\phi_0(\cdot)$ and $\Phi(\cdot,1)=\phi_1(\cdot)$. Since A is a compact almost Kaehler manifold, we have by Stokes' theorem

$$
\begin{aligned}
0&=\int_{A\times[0,1]} d(\Phi'\omega^B\wedge(\omega^A)^{m-1})\\
&=\int_A \phi_1'(\omega^B)\wedge(\omega^A)^{m-1}-\int_A \phi_0'(\omega^B)\wedge(\omega^A)^{m-1}.
\end{aligned}
$$

From the above Lemma, we get $K(\phi_1)=K(\phi_0)$. Q.E.D.

The proof of Theorem C: Let $\Phi: M\times[0,1]\to N$ be an L^2-bounded homotopy. Using cut off function λ_r, we get

$$\begin{aligned}(2.2)\qquad &\int_M \lambda_r\cdot\phi_1'(\omega^N)\wedge(\omega^M)^{m-1}-\int_M \lambda_r\cdot\phi_0'(\omega^N)\wedge(\omega^M)^{m-1}\\ &=\int_{M\times[0,1]} d(\lambda_r\cdot\Phi'(\omega^N)\wedge(\omega^M)^{m-1})\\ &=\int_{M\times[0,1]} d\lambda_r\wedge\Phi'\omega^N\wedge(\omega^M)^{m-1}.\end{aligned}$$

Since Φ is an L^2-bounded homotopy, $\Phi'\omega^N\wedge(\omega^M)^{m-1}\in L^1(M\times[0,1])$. In fact,

$$|\Phi'\omega^N\wedge(\omega^M)^{m-1}|\leqq|\omega^N|\cdot|d\Phi|^2\cdot m^{m-1}=n\cdot m^{m-1}\cdot|d\Phi|^2,$$

where $|\cdot|$ is the norms induced by metric tensors on $M\times[0,1]$ and N. Accordingly,

$$\begin{aligned}&\int_{M\times[0,1]} d\lambda_r\wedge\Phi'\omega^N\wedge(\omega^M)^{m-1}\\ &\leqq\int_{M\times[0,1]}(C_1/r)\cdot n\cdot m^{m-1}\cdot|d\Phi|^2 v_g.\end{aligned}$$

Taking the limit as r tends to infinity, we get $K(\phi_1)=K(\phi_0)$ from the above Lemma and (2.2). Q.E.D.

From Theorem C, we have $E'(\phi_t)-E''(\phi_t)=K(\phi_t)$ a constant independent of t for L^2-bounded variation $\phi_t(:=\Phi(\cdot,t))$, so that

$$\partial E'(\phi_t)/\partial t=\partial E''(\phi_t)/\partial t=1/2\cdot\partial E(\phi_t)/\partial t.$$

Using this fact, we easily obtain Theorem A and Theorem B. The following is immediate from the above.

Corollary D. *If maps $\phi,\tilde{\phi}: M\to N$ between complete almost Kaehler manifolds M, N are L^2-bounded homotopic with ϕ holomorphic and $\tilde{\phi}$ anti-holomorphic, then ϕ and $\tilde{\phi}$ are constant.*

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