111. Selberg Trace Formula for Odd Weight. II

By Shigeki AKIYAMA
Department of Mathematics, Kobe University

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This is continued from [0].

§4. **A dimension formula of the space of cusp forms of weight one.** First we consider the space $\mathcal{S}_m(\Gamma, \chi)$ which consists of C^ν valued holomorphic functions satisfying:

$$\begin{cases} (1) & F\,|\,[T]_m = \chi(T)F \qquad \text{for } T\in\Gamma\,; \\ (2) & \displaystyle\int_{\Gamma\backslash H} {}^tF(z)\overline{F(z)}\,y^m dz < \infty, \end{cases}$$

where $F\,|\,[T]_m = F(T\cdot z)j(T,z)^{-m}$. The connection of this space $\mathcal{S}_m(\Gamma,\chi)$ and Selberg eigenspace is given by the next lemma.

Lemma 1.

$$\mathcal{L}_\chi\left(m+2, \frac{m}{2}\left(1+\frac{m}{2}\right)\right) = y^{(m+2)2/} \exp\left(-\sqrt{-1}(m+2)\phi\right)\mathcal{S}_{m+2}(\Gamma,\chi).$$

$$\mathcal{L}_\chi\left(m, \frac{m}{2}\left(1+\frac{m}{2}\right)\right) = y^{-m/2} \exp\left(-\sqrt{-1}m\phi\right)\overline{\mathcal{S}_{-m}(\Gamma,\bar{\chi})}.$$

Lemma 2. *Suppose*

$$\lambda \neq \frac{m}{2}\left(1+\frac{m}{2}\right), \qquad \textit{then } \dim \mathcal{L}_\chi(m,\lambda) = \dim \mathcal{L}_\chi(m+2,\lambda).$$

Using these two lemmas, we can calculate the difference between the dimension of $\mathcal{S}_m(\Gamma,\chi)$ and that of $\mathcal{S}_{2-m}(\Gamma,\chi)$, and induce the explicit dimension formula for $m\geq 2$. In the case of weight one, we have

Theorem 3.

$$\begin{aligned} &\dim \mathcal{S}_1(\Gamma,\chi) - \dim \mathcal{S}_1(\Gamma,\bar{\chi}) \\ &\quad = \sqrt{-1}\sum_{\{R\}} \frac{\mathrm{Tr}\,(\chi(R))}{2^{\#}\Gamma(R)\sin\theta} \\ &\qquad + \sum_{\substack{\alpha_{ij}\neq 0 \\ \text{regular}}} \left(\frac{1}{2}-\alpha_{ij}\right) - \sum_{\substack{\alpha_{ij}\neq -1/2 \\ \text{irregular}}} \alpha_{ij} - \frac{1}{2}\,\mathrm{Tr}\left(\Phi_1\left(\frac{1}{2}\right)\right). \end{aligned}$$

Now we treat the trace formula in a different way. Assume that $h(r) = h(r,s)$ is a meromorphic function of r and s, and the trace formula is analytically continued to the whole s-plane. Let $h(r,s)$ has a pole $s=m/2$ when $r=\sqrt{-\lambda-1/4}$ and $\lambda=(m/2)(m/2-1)$, and $h(r,s)$ is holomorphic at $s=m/2$ whenever $r\neq\sqrt{-\lambda-1/4}$. This situation can be realized by various functions of r and s. Especially we can take the Selberg kernel $(1/(r^2+(s-1/2)^2))-(1/(r^2+\beta^2))$, where $\beta\gg 0$. Let us compare the residues at $s=m/2$ of both sides. If $m\geq 3$, we get the same formula of Theorem 3. In this case, the hyperbolic contribution vanishes because the Selberg zeta function is holomorphic at $s=m/2$. But if $m=1$, we have

Theorem 4.

$$\dim \mathcal{S}_1(\Gamma, \chi)=\frac{1}{2}\operatorname*{ord}_{s=1/2} Z_\Gamma^*(s,\chi)+\sqrt{-1}\sum_{\{R\}}\frac{\mathrm{Tr}\,(\chi(R))}{4^\#\Gamma(R)\sin\theta}$$
$$+\frac{1}{2}\sum_{\substack{\alpha_{ij}\neq 0\\ \text{regular}}}\left(\frac{1}{2}-\alpha_{ij}\right)-\frac{1}{2}\sum_{\substack{\alpha_{ij}\neq -1/2\\ \text{irregular}}}\alpha_{ij}-\frac{1}{4}\,\mathrm{Tr}\left(\Phi_1\left(\frac{1}{2}\right)\right)$$

where "ord" *denotes the order of zeros.*

By Theorems 3 and 4, we get

Theorem 4′.

$$\dim \mathcal{S}_1(\Gamma, \chi)+\dim \mathcal{S}_1(\Gamma, \bar{\chi})=\operatorname*{ord}_{s=1/2} Z_\Gamma^*(s,\chi).$$

This result explains well why the residue of the Selberg type zeta function appears in the dimension formula of [2], [6] and [7]. Comparing trace formulas of different odd weights, we easily get the following.

Theorem 5. *For* $N\geq 1$, *we have*

$$\dim \mathcal{S}_1(\Gamma, \chi)=\dim \mathcal{L}_\chi\left(2N+1, -\frac{1}{4}\right)+\frac{1}{2}\,\mathrm{Tr}\,\Phi_1\left(\frac{1}{2}\right).$$

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Reference

[0] S. Akiyama: Selberg trace formula for odd weight. I. Proc. Japan Acad., 64A, 341–344 (1988).