109. *On the Inequalities of Erdös-Turán and Berry-Esseen. I*

By Petko D. PROINOV
Department of Mathematics, University of Plovdiv, Bulgaria

(Communicated by Shokichi IYANAGA, M. J. A., Dec. 12, 1988)

1. Introduction. The purpose of this note is to present new generalizations of the celebrated inequalities of Erdös-Turán [4: p. 114] on uniform distribution mod 1 and Berry-Esseen [5: p. 285] for the closeness of two distributions. More precisely, we give some upper bounds for the supremum norm $\|f\|$ and the oscillation $[f]$ of a real-valued function f in terms of its modulus of nonmonotonicity and its Fourier-Stieltjes transform. The modulus of nonmonotonicity was introduced by Sendov [9]. For the properties and other applications of this modulus we refer to [9] and [10].

In Section 3, we generalize and improve all previous versions of the Erdös-Turán inequality which are due to Faĭnleĭb [2], Elliott [1], Niederreiter and Philipp [6] and the author [8]. Moreover, Theorem 3 implies the classical Erdös-Turán inequality (see [4: p. 114]) with constant $C=24/\pi^2$, which is better than the previous one ($C=4$) obtained by Niederreiter and Philipp [6].

The results in Section 4 generalize the Berry-Esseen inequality as well as one of its generalizations obtained by Faĭnleĭb [2] (see also Popov [7] for another form of Faĭnleĭb's inequality). We note that another generalization of the Berry-Esseen inequality was obtained by Popov [7].

2. Moduli of nonmonotonicity. Let f be a real-valued function defined on an interval Δ. The modulus of nonmonotonicity of f was defined by Sendov as follows:

$$\mu(f;\delta)=\sup_{\substack{x'\leqq x\leqq x''\\ x''-x'\leqq\delta}}(|f(x)-f(x')|+|f(x)-f(x'')|-|f(x')-f(x'')|),$$

where the supremum is taken over all points x', x'' and x in Δ satisfying the above inequalities. Following Sendov [9] we say that the function f is locally-monotone on Δ if

$$\mu(f;\delta)\longrightarrow 0 \qquad \text{as} \quad \delta\longrightarrow 0+.$$

For the properties of locally-monotone functions one can see [9].

In order to define some subsets of the class of locally-monotone functions, we consider also the following moduli which were defined by Korneĭchuk [3: p. 111] as follows:

$$\omega_+(f;\delta)=\sup_{0\leqq x''-x'\leqq\delta}(f(x'')-f(x'))$$

and

$$\omega_-(f;\delta)=\sup_{0\leqq x''-x'\leqq\delta}(f(x')-f(x'')),$$

where the supremums are taken over all points x' and x'' lying in Δ and satisfying the above inequalities. Also we write

$$\nu(f;\delta)=\min\{\omega_+(f;\delta),\ \omega_-(f;\delta)\}.$$

It is easy to prove that

$$\mu(f;\delta)\leqq\nu(f;\delta)\leqq\omega(f;\delta)\qquad\text{for all }\delta\geqq 0,$$

where $\omega(f;\delta)$ denotes the modulus of continuity of f on the interval Δ.

We say that the function f is *locally-decreasing* on Δ if

$$\omega_+(f;\delta)\longrightarrow 0\qquad\text{as}\quad\delta\longrightarrow 0+.$$

Analogously, we say that f is *locally-increasing* on Δ if

$$\omega_-(f;\delta)\longrightarrow 0\qquad\text{as}\quad\delta\longrightarrow 0+.$$

From the inequality $\mu(f;\delta)\leqq\nu(f;\delta)$, it follows that if f is locally-decreasing or locally-increasing on Δ, then it is locally-monotone on this interval.

The function f is said to satisfy the *one-sided Lipschitz condition* on Δ with constant L if either

$$\omega_+(f;\delta)\leqq L\delta\qquad\text{for all }\delta\geqq 0,$$

or

$$\omega_-(f;\delta)\leqq L\delta\qquad\text{for all }\delta\geqq 0.$$

It is easy to show (see [8]) that if f satisfies a one-sided Lipschitz condition on a closed interval Δ, then it is a function of bounded variation on this interval.

3. Generalization of the Erdös-Turán inequality. For a Riemann-integrable function f on the unit interval [0, 1], we define its Fourier-Stieltjes transform as follows:

$$\hat{f}(h)=\int_0^1 e^{2\pi i h x}df(x)\qquad\text{for }h\in\boldsymbol{Z}.$$

Theorem 1. *Let f be a periodic function with period* 1, *and let it be Riemann-integrable on* [0, 1]. *Then for every positive integer m and every real $a>1$, we have*

$$[f]\leqq(a+1)\mu\left(f;\ \frac{16}{\pi^2(a-1)}\cdot\frac{1}{m}\right)+\frac{2a}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h}-\frac{1}{m}\right)|\hat{f}(h)|.$$

Corollary 1. *Let $\mu(\delta)$ be a monotone increasing function on* $[0,\infty)$ *with $\mu(0+)=0$, and let $(F_N)_1^\infty$ be a sequence of locally-monotone periodic functions with period* 1, *each function of which satisfies the inequality*

$$\mu(F_N;\delta)\leqq\mu(\delta)\qquad\text{for all }\delta\geqq 0.$$

Suppose also that

$$\lim_{N\to\infty}\hat{F}_N(h)=0\qquad\text{for all }h\in\boldsymbol{N}.$$

Then

$$\lim_{N\to\infty}[F_N]=0.$$

It is easy to show that Corollary 1 is a generalization of the sufficient part of the well known (in the theory of uniform distribution mod 1) Weyl-Schoenberg criterion (see [4: Chapter 1, Theorem 7.3]).

Theorem 2. *Let f be as in Theorem* 1. *Then for every positive integer m and every real $a>1$, we have*

$$[f]\leqq(a+1)\nu\left(f;\ \frac{8a}{\pi^2(a-1)}\cdot\frac{1}{m}\right)+\frac{2a}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h}-\frac{1}{m}\right)|\hat{f}(h)|.$$

Corollary 2. *Let a function f be Riemann-integrable on* $[0, 1]$, *and* $f(0)=f(1)$. *Then for every positive integer m and every real $a>1$, we have the above inequality but with $2(a+1)$ in place of $(a+1)$.*

Corollory 3. *Let F and G be distributions in* $[0, 1]$. *Then for every real $m>0$, and every real $a>1$,*

$$[F-G] \leqq 2(a+1)\omega\left(\frac{8a}{\pi^2(a-1)}\cdot\frac{1}{m}\right)+\frac{2a}{\pi}\sum_{1\leqq h\leqq m}\frac{|\hat{F}(h)-\hat{G}(h)|}{h},$$

where $\omega(\delta)=\min\{\omega(F;\delta), \omega(G;\delta)\}$.

This latter corollary improves in various ways a result of Elliott [1: Theorem 2] and a result of Faĭnleĭb [2: Theorem 3]. For example, if we apply Corollary 3 with $a=\pi^2/(\pi^2-4)$, then we get the estimate

$$[F-G] \leqq 11\omega\left(\frac{1}{m}\right)+\frac{4}{\pi}\sum_{1\leqq h\leqq m}\frac{|\hat{F}(h)-\hat{G}(h)|}{h},$$

which is a refinement of the above mentioned Faĭnleĭb's theorem.

Theorem 3. *Let a function f satisfy the one-sided Lipschitz condition on* $[0, 1]$ *with constant L, and let* $f(0)=f(1)$. *Then for every positive integer m, we have*

$$[f] \leqq \frac{24}{\pi^2}\cdot\frac{L}{m}+\frac{4}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h}-\frac{1}{m}\right)|\hat{f}(h)|.$$

This theorem was proved in [8] with constant 4 in place of $24/\pi^2$. In [8] we noticed that a result of Niederreiter and Philipp [6: Theorem 1′] is a consequence of Theorem 3.

4. Generalization of the Berry-Esseen inequality. It is well known that the Erdös-Turán inequality can be regarded as a discrete analogue of the Berry-Esseen inequality. In this section, we give two theorems which can be regarded as continuous analogues of Theorems 1 and 2, respectively.

For a function f of bounded variation on $(-\infty, \infty)$, we define its *Fourier-Stieltjes transform* as follows:

$$\hat{f}(t)=\int_{-\infty}^{\infty} e^{itx}df(x) \qquad \text{for all real } t.$$

Theorem 4. *Let f be a function of bounded variation on* $(-\infty, \infty)$, *and let* $f(-\infty)=f(\infty)=0$. *Then for every real $T>0$ and every real $a>1$, we have*

$$\|f\| \leqq \frac{a+1}{2}\mu\left(f;\ \frac{32a}{\pi(a-1)}\cdot\frac{1}{T}\right)+\frac{a}{\pi}\int_0^T\left(\frac{1}{t}-\frac{1}{T}\right)|\hat{f}(t)|dt.$$

Theorem 5. *Let f be as in Theorem 4. Then for every real $T>0$ and every real $a>1$, we have*

$$\|f\| \leqq \frac{a+1}{2}\nu\left(f;\ \frac{16a}{\pi(a-1)}\cdot\frac{1}{T}\right)+\frac{a}{\pi}\int_0^T\left(\frac{1}{t}-\frac{1}{T}\right)|\hat{f}(t)|dt.$$

Corollary 4. *Let F and G be distribution functions (on the whole real line). Then for every real $T>0$ and every real $a>1$, we have*

$$\|F-G\| \leqq \frac{a+1}{2}\omega\left(\frac{16a}{\pi(a-1)}\cdot\frac{1}{T}\right)+\frac{a}{\pi}\int_0^T\frac{|\hat{F}(t)-\hat{G}(t)|}{t}dt,$$

where $\omega(\delta)$ is defined as in Corollary 3.

Setting in this latter corollary $a=g\pi/(g\pi-16)$ we obtain

$$\|F-G\|\leqq 15\omega\left(\frac{1}{T}\right)+\frac{3}{\pi}\int_0^T \frac{|\hat{F}(t)-\hat{G}(t)|}{t}dt,$$

which without specified constants is due to Faĭnleĭb [2: Theorem 1] (see also Popov [7: Theorem C]).

Acknowledgement. The author is indebted to Prof. Vasil A. Popov for very stimulating discussions.

(to be continued.)

References

[1] P. D. T. A. Elliott: On distribution functions (mod 1). Quantitative Fourier inversion. J. Number Theory, 4, 509–522 (1972).
[2] A. S. Faĭnleĭb: A generalization of Esseen's inequality and its application in probabilistic number theory. Izv. Akad. Nauk SSSR, Ser. Mat., **32**, 859–879 (1968) (in Russian).
[3] N. P. Korneĭchuk: Exact Constants in Approximation Theory. Moscow (1987) (in Russian).
[4] L. Kuipers and H. Niederreiter: Uniform Distribution of Sequences. Wiley (1974).
[5] M. Loève: Probability Theory. 3rd ed., Princeton (1963).
[6] H. Niederreiter and W. Philipp: Berry-Esseen bounds and a theorem of Erdös and Turán on uniform distribution mod 1. Duke Math. J., **40**, 633–649 (1973).
[7] V. A. Popov: On the inequalities of Berry-Esseen and V. M. Zolotarev. Lect. Notes in Math., vol. 1233, Springer, pp. 114–124 (1987).
[8] P. D. Proinov: On the Erdös-Turán inequality on uniform distribution. I, II. Proc. Japan Acad., **64**A, 27–28, 49–52 (1988).
[9] Bl. Sendov: On some linear methods for approximation of periodic functions in Hausdorff metric. Annuaire Univ. Sofia Fac. Math., **58**, 107–140 (1965) (in Bulgarian).
[10] ——: Some problems of approximation theory of functions and sets in Hausdorff metric. Uspehi Mat. Nauk, **24**, 147–178 (1969) (in Russian).