108. *Two-Phase Stefan Problems for Parabolic-Elliptic Equations*

By Toyohiko AIKI
Department of Mathematics, Graduate School of Sciences,
Chiba University

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1. Statement of the problem. Let us consider a two-phase Stefan problem described as follows: Find a function $u=u(t,x)$ on $Q=(0,T)\times(0,1)$, $0<T<\infty$, and a curve $x=l(t)$, $0<l<1$, on $[0,T]$ such that

$$(0.1)\quad \rho(u)_t - a(u_x)_x + h(t,x) = \begin{cases} f_0 & \text{in } Q_l^+, \\ f_1 & \text{in } Q_l^-, \end{cases}$$

$$h(t,x)\in g(u(t,x)) \quad \text{for a.e. } (t,x)\in Q,$$

$$Q_l^+=\{(t,x)\,;\,0<t<T,\ 0<x<l(t)\},\ Q_l^-=\{(t,x)\,;\,0<t<T,\ l(t)<x<1\},$$

$$(0.2)\quad \begin{cases} u(t,l(t))=0 \quad \text{for } 0\leqq t\leqq T, \\ l'(t)=-a(u_x(t,l(t)-))+a(u_x(t,l(t)+)) \quad \text{for a.e. } t\in(0,T),\ l(0)=l_0, \end{cases}$$

$$(0.3)\quad \rho(u(0,x))=v_0(x) \quad \text{for } 0\leqq x\leqq 1,$$

$$(0.4)\quad \begin{cases} a(u_x(t,0+))\in\partial b_0^t(u(t,0)) & \text{for a.e. } t\in(0,T), \\ -a(u_x(t,1-))\in\partial b_1^t(u(t,1)) & \text{for a.e. } t\in(0,T), \end{cases}$$

where $\rho\colon R\to R$ is a non-decreasing function and $a\colon R\to R$ is a continuous function; $g(\cdot)$ is a maximal monotone graph in $R\times R$; f_0, f_1 are functions on Q; l_0 is a number with $0<l_0<1$ and v_0 is a function on the interval $(0,1)$; for $i=0,1$, b_i^t is a proper l.s.c. convex function on R and ∂b_i^t is its subdifferential. We note that the expression (0.4) includes various boundary conditions such as Dirichlet, Neumann and Signorini boundary conditions.

In the case when $a(r)=r$ and $g(r)\equiv 0$, Crowley [2] proved the uniqueness of solution to the multi-dimensional problem in a weak formulation and recently Cannon-Yin [1] established an existence result for (0.1)–(0.4) under the additional restriction that ρ is strictly increasing in R.

In this paper, we suppose that ρ is non-decreasing, and we are very interested in the additional heat source term $g(u)$, which causes unusual behavior of the solution $\{u,l\}$. For instance, as is seen from the following example, $\Omega_0(t):=\{x\in[0,1]\,;\,u(t,x)=0\}$ has positive linear measure. This region $\Omega_0(t)$ is called the mushy region and was analized by M. Bertsch, P. de Mottoni and L. A. Peletier [1, 2].

Example. Suppose that $T=3$,

$$\rho(r)=\begin{cases} r-1 & \text{for } r>1, \\ 0 & \text{for } |r|\leqq 1, \\ r+1 & \text{for } r<-1, \end{cases} \qquad a(r)=r,$$

$$g(r)=\text{sign}(r)=\begin{cases} 1 & \text{for } r>0, \\ [-1,1] & \text{for } r=0, \\ -1 & \text{for } r<0, \end{cases} \qquad f_0=f_1=0,$$

$$b_i^t(r)=\begin{cases}0 & \text{if } r=g_i(t),\\ \infty & \text{if } r\neq g_i(t),\end{cases}\quad (i=0,1)$$

where

$$g_0(t)=\begin{cases}\frac{1}{2}\left(\frac{1}{4}t+\frac{1}{4}\right)^2 & \text{for } 0\leqq t\leqq 1,\\ \frac{1}{8} & \text{for } 1<t\leqq 2,\\ \frac{1}{2}\left(\frac{1}{4}t-1\right)^2 & \text{for } 2<t\leqq 3,\end{cases}\quad g_1(t)=-g_0(t) \quad \text{for } 0\leqq t\leqq 3,\ l_0=\frac{1}{2},$$

and

$$v_0(x)=\begin{cases}\frac{1}{2}\left(x-\frac{1}{4}\right)^2 & \text{for } x\in\left[0,\frac{1}{4}\right],\\ 0 & \text{for } x\in\left(\frac{1}{4},\frac{3}{4}\right),\\ -\frac{1}{2}\left(x-\frac{3}{4}\right)^2 & \text{for } x\in\left[\frac{3}{4},1\right].\end{cases}$$

Then

$$u(t,x)=\begin{cases}\frac{1}{2}\left\{x-\left(\frac{1}{4}t+\frac{1}{4}\right)\right\}^2 & \text{for } (t,x)\in[0,1]\times\left[0,\frac{t}{4}+\frac{1}{4}\right],\\ 0 & \text{for } (t,x)\in[0,1]\times\left(\frac{t}{4}+\frac{1}{4},-\frac{t}{4}+\frac{3}{4}\right),\\ -\frac{1}{2}\left\{x-\left(-\frac{1}{4}t+\frac{3}{4}\right)\right\}^2 & \text{for } (t,x)\in[0,1]\times\left[-\frac{t}{4}+\frac{3}{4},1\right],\\ \frac{1}{2}\left(x-\frac{1}{2}\right)^2 & \text{for } (t,x)\in(1,2]\times\left[0,\frac{1}{2}\right],\\ -\frac{1}{2}\left(x-\frac{1}{2}\right)^2 & \text{for } (t,x)\in(1,2]\times\left(\frac{1}{2},1\right],\\ \frac{1}{2}\left\{x-\left(-\frac{1}{4}t+1\right)\right\}^2 & \text{for } (t,x)\in(2,3]\times\left[0,-\frac{1}{4}t+1\right],\\ 0 & \text{for } (t,x)\in(2,3]\times\left(-\frac{1}{4}t+1,\frac{1}{4}t\right),\\ -\frac{1}{2}\left(x-\frac{1}{4}t\right)^2 & \text{for } (t,x)\in(2,3]\times\left[\frac{1}{4}t,1\right],\end{cases}$$

$$l(t)=\frac{1}{2}\quad \text{for } 0\leqq t\leqq 3,$$

give a solution of our Stefan problem. In this example, $\Omega_0(t)=\{x\in[0,1]\,;\ u(t,x)=0\}$ has positive linear measure for $t\in[0,1)\cup(2,3]$ and reduces to one point for $t\in[1,2]$.

2. Main results. We begin with the precise assumptions (a1)–(a4) on ρ, a, g, and f_i, b_i^t, $i=0,1$, v_0, under which Stefan problem (0.1)–(0.4) is discussed,

(a1) $\rho: R\to R$ is a Lipschitz continuous and non-decreasing function with $\rho(0)=0$.

(a2) $a: R \to R$ is a continuous function such that

$$a_0|r|^p \leqq a(r)r \leqq a_1|r|^p \qquad \text{for any } r \in R,$$
$$a_0(r-r')^{p-1} \leqq a(r)-a(r') \qquad \text{for any } r, r' \in R, r \geqq r',$$

where a_0 and a_1 are positive constants and $2 \leqq p < \infty$.

(a3) $g(\cdot)$ is a maximal monotone graph in $R \times R$ and $g = \partial\hat{g}$ in R, where $\hat{g}: R \to R$ is a Lipschitz continuous, convex and non-negative function on R with $\hat{g}(0)=0$ and $\partial\hat{g}$ denotes its subdifferential in R.

(a4) For $i=0, 1$ and each $t \in [0, T]$, b_i^t is a proper l.s.c. convex function on R which satisfies the following condition (*) for given functions $\alpha_0 \in W^{1,2}(0, T)$, $\alpha_1 \in W^{1,1}(0, T)$:

(*) For any $0 \leqq s \leqq t \leqq T$ and $r \in D(b_i^s) \equiv \{r \in R ; b_i^s(r) < \infty\}$ there exists $r' \in D(b_i^t)$ such that

$$|r'-r| \leqq |\alpha_0(t)-\alpha_0(s)|(1+|r|+|b_i^s(r)|^{1/p}),$$
$$b_i^t(r')-b_i^s(r) \leqq |\alpha_1(t)-\alpha_1(s)|(1+|r|^p+|b_i^s(r)|).$$

Furthermore for b_i^t, f_i, $i=0, 1$, v_0, l_0, we assume that

(a5-1) $\partial b_0^t(r) \subset (-\infty, 0]$ for any $r<0$ and $t \in [0, T]$, and $\partial b_1^t(r) \subset [0, \infty)$ for any $r>0$ and $t \in [0, T]$;

(a5-2) $f_0, f_1 \in W^{1,2}(0, T; L^2(0, 1)) \cap L^1(0, T; L^\infty(0, 1))$, $f_0 \geqq 0$, $f_1 \leqq 0$ a.e. on Q.

(a5-3) $0<l_0<1$ and there is a function $u_0 \in W^{1,p}(0, 1)$ such that $u_0(i) \in D(b_i^0)$, for $i=0, 1$ and $u_0 \geqq 0$ on $[0, l_0]$, $u_0 \leqq 0$ on $[l_0, 1]$, $v_0 = \rho(u_0)$.

Now we denote by $P=P(b_0^t, b_1^t; g; f_0, f_1; v_0; l_0)$ the system (0.1)–(0.4) and say that a pair $\{u, l\}$ is a solution of P on $[0, T]$, if the following properties (i)–(iii) are fulfilled:

(i) $\rho(u) \in W^{1,2}(0, T; L^2(0, 1))$, $u \in L^\infty(0, T; W^{1,p}(0, 1))$
$l \in W^{1,2}(0, T) (\subset C([0, T]))$ with $0<l<1$ on $[0, T]$;

(ii) (0.1) holds in the sense of $\mathscr{D}'(Q_l^+)$ and $\mathscr{D}'(Q_l^-)$ for some $h \in L^2(Q)$ with $h(t, x) \in g(u(t, x))$ for a.e. $(t, x) \in Q$, and (0.2) and (0.3) are satisfied.

(iii) $b_i^{(\cdot)}(u(\cdot, i))$ is bounded on $[0, T]$, $u(t, i) \in D(\partial b_i^t)$ for a.e. $t \in [0, T]$, $i=0, 1$, and (0.4) holds.

The main results of the present paper are stated as follows:

Theorem 1. *Suppose that assumptions* (a1)–(a5) *hold. Then there exists T_0 with $0<T_0 \leqq T$ such that problem P has at least one solution $\{u, l\}$ on* $[0, T_0]$.

Theorem 2. *Let ρ and a be functions satisfying* (a1) *and* (a2) *respectively, and let $P=P(b_0^t, b_1^t; g; f_0, f_1; v_0, l_0)$ and $\bar{P}=P(\bar{b}_0^t, \bar{b}_1^t; \bar{g}; \bar{f}_0, \bar{f}_1; \bar{v}_0, \bar{l}_0)$ be Stefan problems, where Stefan data of P and $\bar{P}$ are supposed to satisfy conditions* (a3)–(a5). *Further suppose that*

$$\begin{cases} (r'-\bar{r}')(r-\bar{r})^+ \geqq 0 \textit{ for and } r \in D(\partial b_i^t),\ \bar{r} \in D(\partial \bar{b}_i^t), \\ \quad r' \in \partial b_i^t(r),\ \bar{r}' \in \partial \bar{b}_i^t(\bar{r}), i=0, 1, \textit{ and } t \in [0, T]; \end{cases}$$

$$\begin{cases} (r'-\bar{r}')(r-\bar{r})^+ \geqq 0 \textit{ for any } r, \bar{r} \in R, \\ \quad r' \in g(r),\ \bar{r}' \in \bar{g}(\bar{r}),\ f_0 \leqq \bar{f}_0,\ f_1 \leqq \bar{f}_1 \textit{ a.e. on } Q. \end{cases}$$

Let $\{u, l\}$ and $\{\bar{u}, \bar{l}\}$ be solutions of P and $\bar{P}$ on $[0, T]$, respectively. Then, we have for any $0 \leqq s \leqq t \leqq T$

$$(2.1)\qquad \begin{aligned} &|[\rho(u(t))-\rho(\bar{u}(t))]^+|_{L^1(0,1)}+(l(t)-\bar{l}(t))^+ \\ &\leqq\{|[\rho(u(s))-\rho(\bar{u}(s))]^+|_{L^1(0,1)}+(l(s)-\bar{l}(s))^+\} \\ &\times\exp\left\{\int_s^t(|f_0(\tau)|_{L^\infty(0,1)}+|\bar{f}_1(\tau)|_{L^\infty(0,1)})d\tau\right\}. \end{aligned}$$

Corollary. *Under the same assumptions as in Theorem* 1, *problem* P *has at most one solution.*

3. Sketch of the proofs. For $0<2\delta<l_0<1-2\delta$ and $L>0$ we put

$$K(T)=\{l\in C([0,T]): \delta\leqq l(t)\leqq 1-\delta,\ |l'|_{L^2(0,T)}\leqq L,\ l(0)=l_0\}.$$

For any $l\in K(T)$ we denote by $CP(l)$ the following initial-boundary value problem formulated in the non-cylindrical domains Q_l^+ and Q_l^-:

$$\begin{aligned} &\rho(u)_t-a(u_x)_x+h(t,x)=\begin{cases} f_0 & \text{in } Q_l^+, \\ f_1 & \text{in } Q_l^-, \end{cases} \\ &h\in L^2(Q),\ h(t,x)\in g(u(t,x)) \quad \text{for a.e. } (t,x)\in Q, \\ &\rho(u(0,x))=v_0(x) \quad \text{for } 0\leqq x\leqq 1, \\ &u(t,l(t))=0 \quad \text{for } 0\leqq t\leqq T, \\ &a(u_x(t,0+))\in\partial b_0^t(u(t,0)) \quad \text{for a.e. } t\in[0,T] \\ &-a(u_x(t,1-))\in\partial b_1^t(u(t,1)) \quad \text{for a.e. } t\in[0,T]. \end{aligned}$$

The existence and uniqueness of solution to $CP(l)$ were obtained by Kenmochi-Pawlow [6; Theorems 1.1, 1.2]. Using the solution u^l to $CP(l)$ for each l in $K(T)$, we define a mapping $N: K(T)\rightarrow C([0,T])$ by

$$[Nl](t)=l_0-\int_0^t a(u_x^l(s,l(s)-))ds+\int_0^t a(u_x^l(s,l(s)+))ds.$$

By virtue of Kenmochi-Pawlow [6; Theorem 1.4], we see that $N: K(T)\rightarrow C([0,T])$ is a continuous mapping with respect to the topology of $C([0,T])$. Also, for sufficiently small $T_0>0$, N maps $K(T_0)$ into itself. It is obvious that $K(T_0)$ is non-empty, compact and convex in $C([0,T_0])$. Therefore by a well-known fixed point theorem, there is l in $K(T_0)$ such that $Nl=l$. Clearly the pair $\{u^l, l\}$ gives a solution to P on $[0,T_0]$ which has the required properties. Thus we have Theorem 1. Also, Theorem 2 can be derived by using a uniqueness result in [5].

References

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