107. On the Asymptotic Property of the Ordinary Differential Equation

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1. Introduction. In this paper we consider the asymptotic property of the zero solution of the ordinary differential equation

$$x'=f(t,x), \qquad f(t,0)=0 \tag{1}$$

where x and f belong to the n-dimensional real space $\boldsymbol{R}^n$ with Euclidean norm $\|\cdot\|$, t is a real scalar and f is defined and continuously differentiable on $I\times\boldsymbol{R}^n$, $I=[0,\infty)$.

By Liapunov's direct method, Marachkoff proved the theorem for the asymptotic stability of the zero solution of (1) ([1]). He assumed that the derivative of the Liapunov's function with respect to (1): $V'_{(1)}(t,x)$ is negative definite. Some extensions of this condition were given by many authors ([2], [3] etc.). By using a second Liapunov's function $W(t,x)$, Matorosov extended Marachkoff's theorem in the case that $V'_{(1)}(t,x)$ is not negative definite ([4]).

The purpose of this paper is to extend some results related to an asymptotic property of the solution of (1) by means of a second Liapunov's function similar to that of Matorosov. Moreover we attempt to extend other conditions in Marachkoff's theorem. So we obtain some extensions of the above theorems.

2. Notations and definitions. We need some notations and definitions to state our results. If $x, y\in\boldsymbol{R}^n$, we denote the distance of x and y by $d(x,y)=\|x-y\|$. We denote by $E(V^*=0)$ the set $\{x\in\boldsymbol{R}^n : V^*(x)=0\}$, and denote by CIP the families of continuous strictly increasing, positive definite functions.

Definition 1. A function $\phi(t)$ is said to be integrally positive if $\int_S \phi(t)dt=+\infty$ holds on every set $S=\bigcup_{m=1}^{\infty}[\alpha_m,\beta_m]$ such that $\alpha_m<\beta_m<\alpha_{m+1}$, $\beta_m-\alpha_m\geq\delta>0$.

Definition 2. $W'_{(1)}(t,x)$ is said to be strictly not equal to zero in the set $E(V^*=0)$, if for any number α and A it is possible to find a number $r_1(\alpha,A)$ and a continuous functions $\xi(t)$ such that

$$\xi(t)>0, \quad \int_t^{\infty}\xi(s)ds=\infty \quad \text{for any } t$$

and in the set $\{(t,x): \alpha<\|x\|<A,\ d(x,E(V^*=0))<r_1,\ t\geq 0\}$

$$\|W'_{(1)}(t,x)\|\geq\xi(t)>0.$$

Definition 3. $W'_{(1)}(t,x)$ is said to be definitely not equal to zero in the set $E(V^*=0)$, if for any number α and A it is possible to find a number $r_1(\alpha, A)$, $\xi(\alpha, A)$ such that $\|W'_{(1)}(t, x)\|\geq\xi>0$ in the set $\{(t, x): \alpha<\|x\|<A, d(x, E(V^*=0))<r_1, t\geq 0\}$.

Definition 4. $W(t, x)$ admits a higher limit, infinitely small in the set $E(V^*=0)$, if $W(t, x)=0$ in the set $\{(t, x): x\in E(V^*=0), t\geq 0\}$ and if for any small number l, α, A it is possible to find a number $r'(\alpha, A)$ such that $\|W(t, x)\|<l$ in the set $\{(t, x): \alpha<\|x\|<A, d(x, E(V^*=0))<r', t\geq 0\}$

Definition 5. A non-negative function $r(t)$ is said to be diminishing, if $\limsup_{t\to\infty}\int_t^{t+1} r(s)ds=0$.

3. Theorems. Theorem 1. *Suppose that there exist two real-valued functions $V(t, x)$ and $W(t, x)$ which are defined and continuously differentiable on $I\times \boldsymbol{R}^n$. Assume that V and W satisfy the following conditions;*

1) $a(\|x\|)\leq V(t, x)\leq b(\|x\|)$ *in* $I\times \boldsymbol{R}^n$, *where* $a(\cdot), b(\cdot)\in \mathrm{CIP}$, $a(r)\to\infty$ $(r\to\infty)$.
2) $V'_{(1)}(t, x)\leqq -V^*(x)\phi(t)+\psi(t)$ *in* $I\times \boldsymbol{R}^n$, *where* $V^*(x)$ *is a continuous non-negative function,* $\phi(t)$ *is integrally positive, and* $\psi(t)$ *is integrable.*
3) *There exists an* $L>0$ *such that* $\|W(t, x)\|<L$ *in* $I\times \boldsymbol{R}^n$.
4) $W'_{(1)}(t, x)$ *is strictly not equal to zero in the set* $E(V^*=0)$.

If, moreover, the function $f(t, x)$ satisfies the following condition

5) *For any compact set* $M\subset \boldsymbol{R}^n$

$$\|f(t, x)\|\leq N+r(t) \qquad \text{for } x\in M$$

where N is a positive constant depending on M, $r(t)$ is a non-negative function depending on M and diminishing.

Then the zero solution of (1) *is globally equi-attractive.*

Proof of Theorem 1. Let $x(t)=x(t; t_0, x_0)$ be a solution of (1) passing through $(t_0, x_0)\in I\times \boldsymbol{R}^n$. Since $V(t, x)$ is continuous, $V(t_0, x(t_0))<\infty$. By 2)

$$V(t, x(t))=V(t_0, x(t_0))+\int_{t_0}^t V'(s, x(s))ds \tag{2}$$

$$\leq V(t_0, x(t_0))-\int_{t_0}^t V^*(x(s))\phi(s)ds+\int_{t_0}^t \psi(s)ds<\infty.$$

From 1) we can choose an A_0 so large that $V(t, x(t))<a(A_0)$ for any $t\geq t_0$. Thus we have $\|x(t)\|<A_0$, hence every solution is defined on $[t_0, \infty)$.

By (2) we have

$$\int_{t_0}^t V^*(x(s))\phi(s)ds<\infty. \tag{3}$$

By 4) for any α and an A_0, there exist a positive number r_1 and a continuous function $\xi(t)$ such that $\|W'_{(1)}(t, x)\|\geq\xi(t)$ in the set

$$U=\{(t, x): \alpha<\|x\|<A_0, d(x, E(V^*=0))<r_1, t\geq 0\}.$$

Suppose that $x(\tau)\in U$. As long as the solution $x(t)$ remains in the set U in the interval $[\tau, t]$ $(t>\tau)$,

$$2L\geq\|W(t, x(t))\|+\|W(\tau, x(\tau))\|\geq\|W(t, x(t))-W(\tau, x(\tau))\|$$

$$\geq\left\|\int_\tau^t W'(s, x(s))ds\right\|=\left|\int_\tau^t \|W'(s, x(s))\|ds\right|$$

(because of 4) and $x(t) \in U$)

$$\geq \int_\tau^t \xi(s)ds.$$

But $\int_\tau^t \xi(s)ds \to \infty$ $(t \to \infty)$, we can conclude that $x(t)$ cannot stay permanently in the set U. Thus there exists a number T_0 such that $x(t) \notin U$ for some $t > \tau + T_0$.

Next we suppose that $\|x(t)\| > \alpha$ for any t. If at an instant τ_1 $d(x(\tau_1), E(V^*=0)) \leq r_1/2$, it is possible to find α_1, β_1 such that

$$\tau_1 \leq \alpha_1 < \beta_1 \leq \tau_1 + T_0, \qquad d(x(\alpha_1), E(V^*=0)) = r_1/2,$$
$$r_1/2 \leq d(x(t_1), E(V^*=0)) \leq r_1 \qquad \text{for } \alpha_1 \leq t \leq \beta_1,$$
$$d(x(\beta_1), E(V^*=0)) = r_1.$$

Because $x(t)$ cannot stay permanently in the set U. Moreover we can find a $\tau_2 > \beta_1$ such that $d(x(\tau_2), E(V^*=0)) \leq r_1/2$. In fact if there exists an instant t^* such that $d(x(t). E(V^*=0)) > r_1/2$ for any $t \geq t^*$, it is possible to find a positive constant κ such that $V^*(x(t)) > \kappa$. Therefore

$$\int_{t_0}^\infty V^*(x(s))\phi(s)ds > \kappa \int_{t^*}^\infty \phi(s)ds = \infty.$$

This contradicts (3). Thus in the same way we can find $\{[\alpha_m, \beta_m]\,;\, m=1, 2, 3, \cdots\}$ such that

$$\tau_m \leq \alpha_m < \beta_m \leq \tau_m + T_0, \qquad d(x(\alpha_m), E(V^*=0)) = r_1/2,$$
$$r_1/2 \leq d(x(t_m), E(V^*=0)) \leq r_1 \qquad \text{for } \alpha_m \leq t \leq \beta_m,$$
$$d(x(\beta_m), E(V^*=0)) = r_1.$$

Moreover we shall prove that there exists a $\delta > 0$ such that $\beta_m - \alpha_m \geq \delta$. Suppose that $\beta_m - \alpha_m \to 0$ $(m \to \infty)$. By 5)

$$\|x(\beta_m) - x(\alpha_m)\| \leq \int_{\alpha_m}^{\beta_m} \|x'(s)\| ds = \int_{\alpha_m}^{\beta_m} \|f(s, x(s))\| ds$$
$$\leq N(\beta_m - \alpha_m) + \int_{\alpha_m}^{\beta_m} r(s)ds \to 0 \qquad (m \to \infty).$$

On the other hand, $\|x(\beta_m) - x(\alpha_m)\| \geq \|x(\beta_m)\| - \|x(\alpha_m)\| \geq r_1/2$. This is a contradiction. Thus we obtain $\beta_m - \alpha_m \geq \delta > 0$ for any $m \in N$.

Now as long as $t \in [\alpha_m, \beta_m]$, $d(x(t), E(V^*=0)) \geq r_1/2$. Therefore $V^*(x(t)) \geq \kappa$ for any $t \in [\alpha_m, \beta_m]$. Let $S = \bigcup_{m=1}^\infty [\alpha_m, \beta_m]$. Then

$$\int_{\alpha_m}^{\beta_m} V^*(x(s))\phi(s)ds \geq \kappa \int_S \phi(s)ds = \infty.$$

This contradicts (3). Thus we can conclude that there exists an instant $t_1 \geq t_0$ such that $\|x(t_1)\| \leq \alpha$.

Even if there exists an instant $t_1' > t_1$ such that $\|x(t_1')\| > \alpha$, we can find an instant $t_2 > t_1'$ such that $\|x(t_2)\| < \alpha$ in the same way. We shall consider such infinite sequence of instants of time; $\{t_m (m=1, 2, 3, \cdots) : \|x(t_m)\| \leq \alpha\}$. Now by 2) we have

$$V(t, x(t)) \leq V(t_m, x(t_m)) + \int_{t_m}^t \psi(s)ds \qquad (t \geq t_m).$$

Since $\|x(t_m)\| \leq \alpha$, $V(t_m, x(t_m)) \leq b(\|x(t_m)\|) \leq b(\alpha)$. Using the condition that $\psi(t)$ is integrable, $\int_{t_m}^t \psi(s)ds \to 0$ $(t_m \to \infty)$. Thus for any positive number ε

it is possible to find an $\alpha>0$ and $m_0 \in \boldsymbol{N}$ such that for any $m \geq m_0$,

$$V(t, x(t)) \leq V(t_m, x(t_m)) + \int_{t_m}^{t} \psi(s)ds \leq b(\alpha) + \int_{t_m}^{t} \psi(s)ds \leq a(\varepsilon)$$

This implies that $\|x(t)\| \leq \varepsilon$ for any $t \geq t_{m_0}$. Q.E.D.

Theorem 2. *Suppose that there exist two real-valued functions $V(t, x)$ and $W(t, x)$ which are defined and continuously differentiable on $I \times \boldsymbol{R}^n$. Assume that V and W satisfy the following conditions;*

1) *$a(\|x\|) \leq V(t, x)$ in $I \times \boldsymbol{R}^n$, where $a(\cdot) \in \mathrm{CIP}$, $a(r) \to \infty$ $(r \to \infty)$.*
2) *$V'_{(1)}(t, x) < -V^*(x)\phi(t) + \psi(t)$ in $I \times \boldsymbol{R}^n$, where $V^*(x)$ is a continuous non-negative function, $\phi(t)$ is integrally positive, and $\psi(t)$ is integrable.*
3) *$W(t, x)$ admits a higher limit, infinitely small in the set $E(V^*=0)$.*
4) *$W'_{(1)}(t, x)$ is definitely not equal to zero in the set $E(V^*=0)$.*

If, moreover, the function $f(t, x)$ satisfies the following condition

5) *For any compact set $M \subset \boldsymbol{R}^n$*

$$\|f(t, x)\| \leq N + r(t) \qquad \text{for } x \in M$$

where N is a positive constant depending on M, $r(t)$ is a non-negative function depending on M and diminishing.

Then the zero solution of (1) *is globally attractive.*

The proof of Theorem 2 can be given by the same idea as in the proof of Theorem 1.

References

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