105. Kato's Inequality and Essential Selfadjointness for the Weyl Quantized Relativistic Hamiltonian[†]

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1. Introduction. For the nonrelativistic quantum Hamiltonian of a spinless particle of mass m, i.e. the nonrelativistic Schrödinger operator $(1/2m)(-i\partial-A(x))^2$, with magnetic fields, Kato [3] discovered a distributional inequality, which is now called *Kato's inequality*, to attack the problem of essential selfadjointness. The aim of this note is to establish an analogous distributional inequality for the Weyl quantized relativistic Hamiltonian H_A^m with magnetic fields to show the essential selfadjointness of the general Weyl quantized relativistic Hamiltonian

$$H^m=H_A^m+\Phi, \tag{1.1}$$

which corresponds to the classical relativistic Hamiltonian (e.g. [4])

$$h^m(p,x)=h_A^m(p,x)+\Phi(x)\equiv\sqrt{(p-A(x))^2+m^2}+\Phi(x),\quad p\in \boldsymbol{R}^d,\quad x\in \boldsymbol{R}^d. \tag{1.2}$$

Here m is a nonnegative constant. The vector and scalar potentials $A(x)$ $\Phi(x)$ are respectively $\boldsymbol{R}^d$-valued and $\boldsymbol{R}$-valued measurable functions in $\boldsymbol{R}^d$. It is assumed that they satisfy:

$$A(x)\quad \textit{and}\quad \int_{0<|y|<1}|A(x-y/2)-A(x)||y|^{-d}dy\quad \textit{are locally bounded,} \tag{1.3}$$

and

$$\Phi(x)\quad \textit{is in}\quad L^2_{\mathrm{loc}}(\boldsymbol{R}^d)\quad \textit{with}\quad \Phi(x)\geq 0\quad \textit{a.e.} \tag{1.4}$$

For instance, (1.3) is fulfilled by a locally Hölder-continuous $A(x)$.

2. Statement of results. We begin with defining the Weyl quantized relativistic Hamiltonian H_A^m with magnetic fields when $A(x)$ satisfies (1.3). If $A(x)$ is sufficiently smooth and for instance, satisfies

$$|\partial^\alpha A(x)|\leq C_\alpha,\qquad x\in \boldsymbol{R}^d,\ 1\leq|\alpha|\leq N, \tag{2.1}$$

for N sufficiently large, with a constant C_α, then it may be defined as the Weyl pseudo-differential operator $H_A^{m,w}$:

$$(H_A^{m,w}u)(x)=(2\pi)^{-d}\iint_{R^d\times R^d}e^{i(x-y)p}h_A^m\left(p,\ \frac{x+y}{2}\right)u(y)dydp,\quad u\in \mathcal{S}(\boldsymbol{R}^d). \tag{2.2}$$

The integral on the right is an oscillatory integral. Note the condition (2.1) allows the case of constant magnetic fields. The definition of H_A^m for the general $A(x)$ satisfying (1.3) is based on the *Lévy-Khinchin formula* for the conditionally negative definite function $\sqrt{p^2+m^2}$:

$$\sqrt{p^2+m^2}=m-\int_{|y|>0}[e^{ipy}-1-ipyI_{\{|y|<1\}}]n^m(dy). \tag{2.3}$$

Here $I_{\{|y|<1\}}$ is the indicator function of the set $\{|y|<1\}$, and $n^m(dy)$ is the *Lévy*

[†] Dedicated to Professor Shozo KOSHI on the occasion of his sixtieth birthday.

measure which is a σ-finite measure on $\boldsymbol{R}^d\backslash\{0\}$ with $\int_{|y|>0} y^2/(1+y^2)n^m(dy)<\infty$. It is given by

(2.4a) $$n^m(dy)=2(2\pi)^{-(d+1)/2}m^{(d+1)/2}|y|^{-(d+1)/2}K_{(d+1)/2}(m|y|)dy, \qquad m>0,$$

(2.4b) $$n^0(dy)=\pi^{-(d+1)}\Gamma\left(\frac{d+1}{2}\right)|y|^{-(d+1)}dy, \qquad m=0,$$

where $K_\nu(z)$ is the modified Bessel function of the third kind of order ν and $\Gamma(z)$ the gamma function.

Definition. The *Weyl quantized relativistic Hamiltonian* H_A^m corresponding to the symbol $h_A^m(p, x)$ in (1.2) is defined to be the integral operator:

(2.5) $$(H_A^m u)(x)=mu(x)-\int_{|y|>0}[e^{-iyA(x+y/2)}u(x+y)-u(x) -I_{\{|y|<1\}}y(\partial_x-iA(x))u(x)]n^m(dy), \qquad u\in\mathcal{S}(\boldsymbol{R}^d).$$

Of course, H_A^m in (2.5) coincides, on $\mathcal{S}(\boldsymbol{R}^d)$, with $H_A^{m,w}$ in (2.2), if $A(x)$ satisfies (2.1). It is seen that H_A^m defines a linear operator in $L^2(\boldsymbol{R}^d)$ with domain $C_0^\infty(\boldsymbol{R}^d)$, and by the rotational invariance of the Lévy measure $n^m(dy)$ that H_A^m is symmetric, i.e. $(H_A^m\varphi, \psi)=(\varphi, H_A^m\psi)$, $\varphi, \psi\in C_0^\infty(\boldsymbol{R}^d)$. For $u\in L^2(\boldsymbol{R}^d)$ the distribution $H_A^m u$ is defined by $(H_A^m u, \varphi)=(u, H_A^m\varphi)$, $\varphi\in C_0^\infty(\boldsymbol{R}^d)$. It can be shown ([6], [2]) that $H_A^{m,w}$ is essentially selfadjoint on $C_0^\infty(\boldsymbol{R}^d)$, when both $A(x)$ and its derivatives $\partial^\alpha A(x)$ up to sufficiently higher order are continuous and bounded. It has recently been shown by Nagase-Umeda [5] when $A(x)$ satisfies (2.1). The condition (1.3) is suggested by the path integral representation for the semigroup $\exp[-t(H_A^{m,w}-m)]$ obtained in [2] (cf. [1]) which is still valid in this case.

The results of the present note are the following two theorems.

Theorem 1. *Assume that* $A(x)$ *and* $\Phi(x)$ *satisfy* (1.3) *and* (1.4). *Then*:
(i) $H^m=H_A^m+\Phi$, *and, in particular,* H_A^m, *is essentially selfadjoint on* $C_0^\infty(\boldsymbol{R}^d)$. (ii) *The unique selfadjoint extension of* H_A^m, *denoted again by the same* H_A^m, *is bounded from below by* m: $H_A^m\geq m$.

The proof of Theorem 1-(i) will be done just in the same way as in Kato [3], if such a distributional inequality between H_A^m and $\sqrt{-\Delta+m^2}$ as in Theorem 2 below is established. It may be regarded as Kato's inequality for H_A^m. Theorem 1-(ii) will also follow from the proof of Theorem 2.

Theorem 2. *Assume* $A(x)$ *satisfies* (1.3). *If* v *is in* $L^2(\boldsymbol{R}^d)$ *with* $H_A^m v$ *in* $L^1_{\mathrm{loc}}(\boldsymbol{R}^d)$, *then*

(2.6) $$\mathrm{Re}\,[(\mathrm{sgn}\, v)H_A^m v]\geq\sqrt{-\Delta+m^2}\,|v|,$$

in the sense of distributions. Here $\mathrm{sgn}\, v$ *is a bounded function in* $\boldsymbol{R}^d$ *defined by*

$$(\mathrm{sgn}\, v)(x)=\begin{cases}\overline{v(x)}/|v(x)|, & \text{if } v(x)\neq 0,\\ 0\quad, & \text{if } v(x)=0.\end{cases}$$

3. Sketch of proof of Theorem 2. Suppose first that v is C^∞ and L^2. Then $H_A^m v$ is L^2_{loc} and hence L^1_{loc}. Using the expression (2.3) of H_A^m we can show

(3.1) $$\mathrm{Re}\,[(\overline{v(x)}/v_\varepsilon(x))[H_A-m]v]\geq[\sqrt{-\Delta+m^2}-m]v_\varepsilon,$$

where $v_\varepsilon(x)=\sqrt{|v(x)|^2+\varepsilon^2}$, $\varepsilon>0$. Next, in the general case, let $v^\delta=\rho_\delta * v$ where $\rho_\delta(x)=\delta^{-d}\rho(x/\delta)$, $\delta>0$, and $\rho(x)$ is a nonnegative C_0^∞ function with support in the sphere of radius one about the origin in $\boldsymbol{R}^d$ and with $\int \rho(x)dx=1$. Then v^δ is C^∞ and L^2, so that (3.1) holds for v^δ in place of v. Then we tend $\delta \downarrow 0$ first and then $\varepsilon \downarrow 0$ to get (2.6). To prove this part, we need to know that $H_A^m v^\delta \to H_A^m v$ as $\delta \downarrow 0$. To this end we must give a kind of integral representation for the function $v \in L^2$ with $H_A^m v \in L_{\mathrm{loc}}^1$ to show its regularity.

A full account of the present note will be published elsewhere.

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