103. Ultra-hyperbolic Approach to some Multi-dimensional Inverse Problems

By Takashi SUZUKI
Department of Mathematics, Faculty of Science,
Tokyo Metropolitan University

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1. Introduction. Our aim is to extend our previous work [2] and establish some uniqueness results for spectral and evolutional inverse problems of multi-dimensional space variables. Thus, let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and $Pu = \nabla\cdot(a\nabla u) + cu$ be a second order formally self-adjoint uniformly elliptic differential operator with smooth coefficients $a = (a_{ij}(x))$ and $c = c(x)$ on $\bar{\Omega}$. We consider the parabolic initial boundary value problem

(1) $$\frac{\partial u}{\partial t} = Pu \ (\text{in } \Omega\times(0,\infty)), \quad u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial \nu_P}\right|_{\partial\Omega} = F(\xi, t),$$

where $\partial/\partial\nu_P = \sum_{i,j} \nu_i a_{ij}(x)(\partial/\partial x_j)$, $\nu = (\nu_{ij})$ being the outer unit normal vector on $\partial\Omega$. Our concern is to determine the coefficients $a = (a_{ij})$ and c through the boundary input $F = f(\xi, t)$ and output $u = u(\xi, t)$ $(\xi \in \Gamma, 0 < t < T)$, where $T > 0$ and $\Gamma \subset \partial\Omega$ with $|\Gamma| > 0$. Hence let Q be a similar elliptic operator and take the equation

(2) $$\frac{\partial v}{\partial t} = Qv \ (\text{in } \Omega\times(0,\infty)), \quad v|_{t=0} = 0, \quad \left.\frac{\partial v}{\partial \nu_Q}\right|_{\partial\Omega} = F(\xi, t).$$

Then, our uniqueness question is formulated as follows: Does

(3) $$v(\xi, t) = u(\xi, t) \qquad (\xi \in \Gamma, 0 < t < T)$$

imply $Q = P$?

2. Reduction to spectral problems. Let P_N and Q_N be the realizations in $X = L^2(\Omega)$ of the differential operators P and Q under the Neumann boundary conditions $\partial/\partial\nu_P \cdot = \partial/\partial\nu_Q \cdot = 0$, respectively. The eigenvalues and eigenfunctions of $-P_N$ and $-Q_N$ are denoted by $\{\lambda_j\}$, $\{\mu_j\}$ $(-\infty < \lambda_1 < \lambda_2 \leqq \cdots \to +\infty,\ -\infty < \mu_1 < \mu_2 \leqq \cdots \to +\infty)$ and $\{\varphi_j\}, \{\psi_j\}$ $(\|\varphi_j\|_{L^2(\Omega)} = \|\psi_j\|_{L^2(\Omega)} = 1)$, respectively. Then, supposing $F(\xi, t) = h(t)f(\xi)$ with $h \not\equiv 0$, we can deduce (e.g. [2]) from (3) that

(4) $$r(\xi, t) = s(\xi, t) \qquad (\xi \in \Gamma, 0 < t < \infty),$$

where $r(x, t) = \sum_j e^{-t\lambda_j}\varphi_j(x)\int_{\partial\Omega}\varphi_j(\xi)f(\xi)d\sigma_\xi$ and $s(x, t) = \sum_j e^{-\mu_j t}\psi_j(x)\int_{\partial\Omega} x_j(\xi) \cdot f(\xi)d\sigma_\xi$. Taking $F(\xi, t) = F_l(\xi, t) = h_l(t)f_l(\xi)$ with $h_l \not\equiv 0$ for $l \in S$, we suppose the following condition, where $J_\lambda = \{j \mid \lambda_j = \lambda\}$ and $L_\lambda = \{j \mid \mu_j = \lambda\}$ for $\lambda \in \boldsymbol{R}$:

(5) The matrices $(\alpha_{jl})_{j\in J_\lambda, l\in S}$ and $(\beta_{jl})_{j\in L_\lambda, l\in S}$ are both of full-rank when $J_\lambda \neq \phi$ or $L_\lambda \neq \phi$, where $\alpha_{jl} = \int_{\partial\Omega}\varphi_j(\xi)f_l(\xi)d\sigma_\xi$ and $\beta_{jl} = \int_{\partial\Omega}\psi_j(\xi)f_l(\xi)d\sigma_\xi$.

From the first condition of (5) and the asymptotic behavior as $t\to\infty$ of both sides of (4), we can furthermore deduce for $\lambda\in\boldsymbol{R}$, $l\in S$ and $x\in\Gamma$ that

(6) $$J_\lambda=L_\lambda \quad\text{and}\quad \sum_{j\in J_\lambda}\varphi_j(x)\int_{\partial\Omega}\varphi_j(\xi)f_l(\xi)d\sigma_\xi=\sum_{j\in L_\lambda}\psi_j(x)\int_{\partial\Omega}\psi_j(\xi)f_l(\xi)d\sigma_\xi,$$

because $\{\varphi_j\}_{j\in J_\lambda}$ and $\{\psi_j\}_{j\in J_\lambda}$ are linearly independent systems on Γ from $|\Gamma|>0$ and Calderón's uniqueness theorem. Therefore, the relation $\varphi_j(x)=\sum_{k\in J_\lambda}\gamma_{jk}\psi_k(x)\equiv\tilde{\psi}_j(x)$ on $x\in\Gamma$ for $j\in J_\lambda$ holds, where $\{\gamma_{jk}\}$ are real numbers. Hence we recall that $(\beta_{jl})_{j\in L_\lambda,l\in S}$ is full-rank.

Now we suppose the important assumption that $\operatorname{supp} f_l\subset\Gamma$ for each $l\in S$. Then the second relation in (6) reduces to $\sum_{j,m\in J_\lambda}\gamma_{jk}\gamma_{jm}\beta_{ml}=\beta_{kl}$ for $k\in J_\lambda$ and $l\in S$ again by Calderón's theorem so that ${}^T(\gamma_{jk})(\gamma_{jk})=(\delta_{jk})$. Hence $\{\tilde{\psi}_j\}$ $(j\in J_\lambda)$ becomes an L^2-orthonormal system. Taking $\tilde{\psi}_j$ instead of ψ_k, we arrive at

(7) $$\lambda_j=\mu_j \text{ and } \varphi_j(x)=\psi_j(x) \quad\text{for}\quad j\in N \text{ and } x\in\Gamma.$$

3. Isospectral deformation. For given integer m, we take sufficiently large λ and s so that $L_s(x,y;\lambda)=\sum_j\{\psi_j(x)-\varphi_j(x)\}\varphi_j(y)(\lambda_j+\lambda)^{-s}\in C^m(\bar{\Omega}\times\bar{\Omega})$ and $M_s(x,y;\lambda)=\sum_j\psi_j(x)\{\varphi_j(y)-\psi_j(y)\}(\mu_j+\lambda)^{-s}\in C^m(\bar{\Omega}\times\bar{\Omega})$ and put $L(x,y)=(-P_y+\lambda)^sL_s(x,y;\lambda)\in C^m(\bar{\Omega}_x\to\mathcal{D}'_y(\Omega))$ and $M(x,y)=(-Q_x+\lambda)^sM_s(x,y;\lambda)\in C^m(\bar{\Omega}_y\to\mathcal{D}'_x(\Omega))$. Then, L and M are independent of λ and s and the first relations in (7) implies $K(x,y)\equiv L(x,y)=M(x,y)\in C^\infty(\bar{\Omega}_x\to\mathcal{D}'_y(\Omega))\cap C^\infty(\bar{\Omega}_y\to\mathcal{D}'_x(\Omega))$[1] as two elements in $\mathcal{D}'(\Omega\times\Omega)$ as well as

(8) $$\Box K=0 \quad\text{in}\quad \bar{\Omega}\times\Omega\backslash D \quad\text{and}\quad \Omega\times\bar{\Omega}\backslash D,$$

where $\Box=-Q_x+P_Y$ and $D=\{(x,x)\,|\,x\in\Omega\}$ ([2]).

The second relation of (7) implies $L_s=0$ on $\Gamma\times\Omega$ so that $K|_{\Gamma\times\Omega}=0$. On the other hand, the ultra-hyperbolic equation (8) gives $Q_x^mK=P_y^mK$ on $\Gamma\times\Omega$ so that $Q_x^mK|_{\Gamma\times\Omega}=0$ for $0,1,2,\cdots$. However, we have the identity for $0<t<\infty$ that

(9) $$F_t(x,y)=\sum_{m=0}^{\infty}\frac{t^m}{m!}Q_x^mK(x,y),$$

where

$$F_t(x,y)=\sum_j e^{-t\lambda_j}\psi_j(x)\{\varphi_j(y)-\psi_j(y)\}.$$

Namely, in the right-hand side of the first equality, the series converges in $\mathcal{D}'_y(\Omega)$ for each fixed $x\in\bar{\Omega}$ to the smooth function $F_t=F_t(x,y)$ in $y\in\Omega$ given in the second equality.[2] Therefore, $F_t|_{\Gamma\times\Omega}=0$ holds for $0<t<\infty$. Now, comparing the behavior as $t\to\infty$, we can conclude that $\sum_{j\in J_\lambda}\psi_j(x)\{\varphi_j(y)-\psi_j(y)\}=0$ for $\lambda\in\boldsymbol{R}$ and $(x,y)\in\Gamma\times\Omega$, and hence $\varphi_j\equiv\psi_j$ again by Calderón's theorem. Thus, we obtain $P_N=Q_N$ as two operators in X, so that the coefficients of P and Q coincide with each other.

4. Remarks. (i) So far we have proved that (7) implies $P\equiv Q$. This is regarded as a multi-dimensional version of the Gel'fand-Levitan theory [1].

1) Formally, this relation reads as $K(x,y)=\sum_j\{\psi_j(x)-\varphi_j(x)\}\varphi_j(y)=\sum_j\psi_j(x)\{\varphi_j(y)-\psi_j(y)\}=\sum_j\psi_j(x)\varphi_j(y)-\delta(x-y)$.

2) Formally, this relation reads as $\sum_j e^{-t\lambda_j}\psi_j(x)\{\varphi_j(y)-\psi_j(y)\}=\sum_{m,j}(t^m/m!)(-\lambda_j)^m\psi_j(x)\cdot\{\varphi_j(y)-\psi_j(y)\}$.

(ii) Our result suggests that identifiability is guaranteed in most cases when input and output are taken from the same area. Actually, a similar method implies the uniqueness in the parabolic inverse problem

$$\frac{\partial u}{\partial t}=Pu+h(t)f(x) \quad (\text{in } \Omega\times(0, \infty)), \quad u|_{t=0}=0, \quad u|_{\partial\Omega}=0 \tag{10}$$

with the output $u|_{\omega}$, where $\omega\subset\Omega$ is a non-void open subset. Namely, identifiability holds even for this problem under a certain algebraic condition as (5) if $h\not\equiv 0$ and $\operatorname{supp} f\subset\omega$.

References

[1] Gel'fand, I. M. and Levitan, B. M.: On the determination of a differential equation from its spectral function. AMS Transl., (2) **1**, 253–304 (1955) (English translation).

[2] Suzuki, T.: On a multi-dimensional inverse parabolic problem. Proc. Japan Acad., **62**A, 83–86 (1986).