98. *On Unit Groups of Algebraic Number Fields*

By Shin-ichi KATAYAMA
College of General Education, University of Tokushima

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1. Let K be a Galois extension of an algebraic number field k and U_K the unit group of the idele group $K_A^\times$. $O_K^\times$ denotes the global unit group of K. $N_{K/k}$ denotes the norm map from $K_A^\times$ to $k_A^\times$.

In our paper [2], we have proved the following isomorphism

$$(*)\qquad (N_{K/k}^{-1}(1)\cap(U_K\cdot K^\times))/(N_{K/k}^{-1}(1)\cap U_K)(N_{K/k}^{-1}(1)\cap K^\times) \cong (O_k^\times\cap N_{K/k}K^\times)/N_{K/k}O_K^\times.$$

In this paper, this result will be generalized by using the cohomological language.

2. First, we consider the following commutative diagram of cochain complexes with exact rows and columns.

$$(1)\qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & \| & & \\
0 & \longrightarrow & B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & A_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_2} & & & & \\
 & & C_1 & = & C_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Let us denote the connecting homomorphisms derived from (1) by

$$\begin{aligned}
\delta_1 &: H^r(A_3)\longrightarrow H^{r+1}(A_1),\\
\delta_2 &: H^r(A_3)\longrightarrow H^{r+1}(B_1),\\
\gamma_1 &: H^r(C_1)\longrightarrow H^{r+1}(A_1),\\
\gamma_2 &: H^r(C_1)\longrightarrow H^{r+1}(A_2) \qquad (r\in \mathbf{Z}).
\end{aligned}$$

We denote the homomorphism $H^r(A_1)\to H^r(A_2)$ induced from a_1 by the same symbol a_1. The homomorphisms $a_2, b_1, b_2, \varphi_1, \varphi_2, \psi_1, \psi_2$ are defined in a similar way. Then, by the elementary diagram chasing, we have the following lemma.

Lemma. *With the notation as above, we have the following isomorphisms*

$$(2)\qquad H^r(B_2)/(\mathrm{Im}\,\varphi_2+\mathrm{Im}\,b_1)\cong \mathrm{Ker}\,a_1\cap\mathrm{Ker}\,\varphi_1\subset H^{r+1}(A_1),$$

$$(3)\qquad H^r(A_1)/(\mathrm{Im}\,\delta_1+\mathrm{Im}\,\gamma_1)\cong \mathrm{Ker}\,b_2\cap\mathrm{Ker}\,\psi_2\subset H^r(B_2).$$

Proof. Proof of (2). Since $\mathrm{Im}\,b_1=\mathrm{Ker}\,b_2$, we have

$$H^r(B_2)/(\mathrm{Im}\,\varphi_2+\mathrm{Im}\,b_1)\cong b_2(H^r(B_2))/b_2\varphi_2(H^r(A_2)).$$

From the fact that $b_2\varphi_2=a_2$ and $\mathrm{Im}\, a_2=\mathrm{Ker}\,\delta_1$, we have

$$b_2(H^r(B_2))/b_2\varphi_2(H^r(A_2))\cong\delta_1 b_2(H^r(B_2)).$$

Let us show the equality $\delta_1 b_2(H^r(B_2))=\mathrm{Ker}\, a_1\cap\mathrm{Ker}\,\varphi_1$. $a_1(\delta_1 b_2)=(a_1\delta_1)b_2=0$ and $\varphi_1(\delta_1 b_2)=(\varphi_1\delta_1)b_2=\delta_2 b_2=0$. Hence we have

$$\delta_1 b_2(H^r(B_2))\subset\mathrm{Ker}\, a_1\cap\mathrm{Ker}\,\varphi_1.$$

On the other hand, for any $x\in\mathrm{Ker}\, a_1\cap\mathrm{Ker}\,\varphi_1$, there exists an element y of $H^r(A_3)$ such that $\delta_1(y)=x$. From the fact $\varphi_1(x)=0$, we have $\delta_2(y)=\varphi_1\delta_1(y)=\varphi_1(x)=0$. Since $\mathrm{Ker}\,\delta_2=\mathrm{Im}\, b_2$, there exists an element $z\in H^r(B_2)$ such that $y=b_2(z)$. Hence $x=\varphi_1 b_2(z)$. Therefore $\varphi_1 b_2(H^r(B_2))\supset\mathrm{Ker}\, a_1\cap\mathrm{Ker}\,\varphi_1$. In the same way as above, we can easily verify the isomorphism (3).

3. K, k being as above, we denote the Galois group of K/k by G. U_K is the unit group of the idele group $K_A^\times$ and the multiplicative group $K^\times$ is considered to be a subgroup of $K_A^\times$. Then the global unit group $O_K^\times$ is $U_K\cap K^\times$. We denote the principal ideal group of K by P_K. Then we have the following commutative diagram of G-modules with exact rows and columns.

$$(4)\qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & O_K^\times & \xrightarrow{a_1} & K^\times & \xrightarrow{a_2} & P_K & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & \| & & \\
0 & \longrightarrow & U_K & \xrightarrow{b_1} & U_K\cdot K^\times & \xrightarrow{b_2} & P_K & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_2} & & & & \\
 & & U_K/O_K^\times & = & U_K/O_K^\times & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then the cochain complexes derived from this diagram satisfies the assumption of the lemma. From the case (2) for $r=-1$, we have the isomorphism

$$H^{-1}(G, U_K\cdot K^\times)/(b_1(H^{-1}(G, U_K))+\varphi_2(H^{-1}(G, K^\times)))$$
$$\cong\mathrm{Ker}\, a_1\cap\mathrm{Ker}\,\varphi_1\subset H^0(G, O_K^\times).$$

Here $H^{-1}(G, U_K\cdot K^\times)/(b_1(H^{-1}(G, U_K)+\varphi_2(H^{-1}(G, K^\times))$ is isomorphic to $(N_{K/k}^{-1}(1)\cap(U_K\cdot K^\times))/(N_{K/k}^{-1}(1)\cap U_K)(N^{-1}(1)\cap K^\times)$. On the other hand, from the fact that $H^0(G, U_k)\to H^0(G, K_A^\times)$ is injective, we have

$$\mathrm{Ker}\,\varphi_1=\mathrm{Ker}\,(H^0(G, O_K^\times)\longrightarrow H^0(G, U_K))=\mathrm{Ker}\,(H^0(G, O_K^\times)\longrightarrow H^0(G, U_K\cdot K^\times)).$$

Hence

$$\mathrm{Ker}\,\varphi_1\subset\mathrm{Ker}\, a_1=\mathrm{Ker}\,(H^0(G, O_K^\times)\longrightarrow H^0(G, K^\times)).$$

Therefore $\mathrm{Ker}\, a_1\cap\mathrm{Ker}\,\varphi_1=\mathrm{Ker}\, a_1=(O_K^\times\cap N_{K/k}K^\times)/N_{K/k}O_K^\times$ for this case. Hence we have obtained the isomorphism (*). Several similar results and the applications to the number theory will be published elsewhere.

References

[1] K. S. Brown: Cohomology of Groups. Springer-Verlag, Berlin-Heidelberg-New York (1982).

[2] S. Katayama: $E(K/k)$ and other arithmetical invariants for finite Galois extensions (to appear in Nagoya Math. J.).