95. A Note on Isocompact wM Spaces and Mappings

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Introduction. T_2 isocompact wM spaces behave well like T_2 paracompact M spaces. For example, if $f: X \to Y$ is a closed, continuous map of a T_2 isocompact wM space X onto Y, then $Y = \cup_{n \geqslant 0} Y_n$, where, for each $n \geqslant 1$, Y_n is discrete in Y and $f^{-1}(y)$ is compact for each $y \in Y_0$. As such, we investigate some interesting properties of such spaces and their images under nice maps. Refer [5], [1], [4], [2] and [3] respectively, for the notions of q, point countable and countable type, wM, isocompactness, and quasi-G_δ diagonal.

Main section. Theorem 1. (i) *A T_1 space X of point countable type is a q space.* (ii) *A regular isocompact q space X is point countable type.*

Proof of (i). Let $x \in X$ and K be a compact subset of X of countable character with $x \in K$. Let $\{U_n | n \geqslant 1\}$ be a decreasing local base at K. To claim that $\{U_n\}_n$ is a q sequence at x, let $x_n \in U_n$ for each n. Suppose $\{x_n\}_n$ does not cluster. Then, $D = \{x_n | n \geqslant 1\}$ is closed. Assume $K \cap D = \emptyset$. Then, $X - D$ is an open nhd of K. Since, $U_n \not\subset X - D$ for each n, we have a contradiction.

Proof of (ii). Let $x \in X$ and $\{U_n\}_n$ be a q sequence at x with $\overline{U}_{n+1} \subset U_n$ for each n. Let $C(x) = \cap_n U_n$. It follows that $C(x)$ is of countable character and $x \in C(x)$. Therefore X is of point countable type. Q.E.D.

Theorem 2. *If a regular space X with quasi-G_δ diagonal is a q space or a space of point countable type, then the space is first countable.*

Proof. By the Theorem 1 (i), X is a q space in either case. Let $\{\mathcal{U}_n\}_n$ be a quasi-G_δ diagonal sequence. Let $x \in X$, $\{G_n\}_n$ be a q sequence at x and $\{n_k\}_k$ be the strictly increasing sequence of natural numbers with $x \in St(x, \mathcal{U}_n) = \cup\{U \in \mathcal{U}_n | x \in U\}$, iff $n = n_k$ for some $k \leqslant n$. By induction, we can obtain a sequence $\{H_m\}_m$ of open sets with $x \in H_{m+1} \subset \overline{H}_{m+1} \subset H_m \cap G_{m+1} \cap U_{n_{m+1}}$ for each m, where $x \in U_{n_m} \in \mathcal{U}_{n_m}$. It follows that $\{H_m | m \geqslant 1\}$ is a local base at x. Q.E.D.

Corollary 2.1. *If a T_2 wM space with quasi-G_δ diagonal is a quotient image of a locally compact, separable and metrizable space, then the space is locally compact, separable and metrizable.*

Proof. Apply the Theorem 2 and a result of A. H. Stone [7]. Q.E.D.

Theorem 3. *A T_2 isocompact wM space X is countable type.*

Proof. Let $\{\mathcal{U}_n\}_n$ be a decreasing wM sequence and $K \subset X$ be compact. Let $\mathcal{W}_1$ be a finite subcollection of $\mathcal{U}_1$ with $K \subset W_1 = \cup \mathcal{W}_1$. Let $\mathcal{W}'_2$ be an open collection with $K \subset \cup \mathcal{W}'_2$ such that $\overline{\mathcal{W}}'_2 = \{\overline{W} | W \in \mathcal{W}'_2\}$ refines $\mathcal{W}_1 \wedge \mathcal{U}_2$

$=\{W\cap U \mid W\in \mathcal{W}_1$ and $U\in \mathcal{U}_2\}$. Let $\mathcal{W}_2$ be a finite subcollection of $\mathcal{W}_2'$ with $K\subset W_2=\cup \mathcal{W}_2$. Continuing this way, we can obtain a sequence $\{\mathcal{W}_n\}_n$ of finite open collections with $K\subset W_n=\cup \mathcal{W}_n$ and $\overline{\mathcal{W}}_{n+1}$ refines $\mathcal{W}_n\wedge \mathcal{U}_{n+1}$ for each n. Let $D=\cap_n W_n$. Then $K\subset D$ and D is a compact set of countable character. Q.E.D.

Corollary 3.1. *A T_2 isocompact wM space is a k space.*

By a result of J. E. Vaughan [8], a Tychonoff isocompact wM space is a generalized G_δ set in its compactification and equivalently, its complement in its compactification is Lindelöf.

By a result of H. H. Wicke [9], a T_2 space is point countable type, iff it is an open, continuous image of a T_2 isocompact wM space; a T_1 regular isocompact space is a q space, iff it is an open, continuous image of a T_2 isocompact wM space (in fact, a T_2 paracompact p space).

Theorem 4. *A quotient image of a regular isocompact q space is a k space.*

Proof. Let $f: X\to Y$ be a quotient map of a regular isocompact q space X onto Y. Let $F\subset Y$ be such that $F\cap C$ is closed in C for every compact $C\subset Y$. To claim that F is closed in Y, we prove that $f^{-1}(F)$ is closed in X. Suppose $x\in \overline{f^{-1}(F)}-f^{-1}(F)$. Let $\{U_n\}_n$ be a q sequence at x with $\overline{U}_{n+1}\subset U_n$ for each n and $C(x)=\cap_n U_n$. Then, $C(x)$ is compact. Let $f(x)=y$.

(I) Suppose $x\in \overline{C(x)\cap f^{-1}(F)}$. For any open nhd W of y, $f^{-1}(W)\cap C(x)\cap f^{-1}(F)\neq\emptyset$. Therefore $y\in \overline{f(C(x))\cap F}$. Since $x\notin f^{-1}(F)$, we have $x\in C(x)\cap(X-f^{-1}(F))$, which implies $y\in f(C(x))\cap(Y-F)$. Therefore $f(C(x))\cap F$ is not closed in $f(C(x))$, which is a contradiction to the definition of F.

(II) Suppose $x\notin \overline{C(x)\cap f^{-1}(F)}$. There is an open nhd U of x with $\overline{U}\cap C(x)\cap f^{-1}(F)=\emptyset$. Let $V_n=U\cap U_n$ for each n, and $x_n\in V_n\cap f^{-1}(F)$ for each n. Let x_0 be a cluster point of the sequence $\{x_n\}_n$. Then $x_0\in C(x)\cap \overline{U}$. Let $K=\overline{\{x_n \mid n\geqslant 1\}}$. Then K is compact, and $x_0\in \overline{K\cap f^{-1}(F)}$. Let $y_0=f(x_0)$. Now $x_0\in K$, $x_0\in C(x)\cap\overline{U}$ and $\overline{U}\cap C(x)\cap f^{-1}(F)=\emptyset$ imply that $x_0\in K\cap(X-f^{-1}(F))$. Therefore $y_0\in f(K)\cap(Y-F)$. If W is an open nhd of y_0, then $f^{-1}(W)\cap K\cap f^{-1}(F)\neq\emptyset$, which implies that $W\cap f(K)\cap F\neq\emptyset$. Therefore $y_0\in\overline{f(K)\cap F}$, which implies that $f(K)\cap F$ is not closed in $f(K)$, which contradicts the definition of F. Therefore $f^{-1}(F)=\overline{f^{-1}(F)}$. Q.E.D.

Corollary 4.1. *A regular isocompact q space is a k space.*

By a result of J. Nagata [6], we have the following corollaries.

Corollary 4.2. *A T_2 space is a k space, iff it is a quotient image of a T_2 isocompact wM space.*

Corollary 4.3. *A T_1 regular isocompact q space is a quotient image of a T_2 paracompact M space.*

Theorem 5. *Let $f: X\to Y$ be a closed, continuous map of a T_2 isocompact wM space X onto Y. Then the following are equivalent.*

(i) *Y is a regular q space.*

(ii) *Y is a regular space of point countable type.*

(iii) *The boundary* $\partial f^{-1}(y)$ *of* $f^{-1}(y)$ *is compact for each* $y \in Y$.

(iv) Y *is a* T_2 *isocompact wM space.*

Proof. By the Theorems 1 and 3, we have (iv)→(ii)→(i). E. Michael has shown that (i)→(iii), [5]. We need to show, now, that (iii)→(iv): For each $y \in Y$, let

$$L(y)=\begin{cases}\partial f^{-1}(y) & \text{if } \partial f^{-1}(y)\neq\emptyset;\\ f^{-1}(y)-\{p_y\}, \text{ where, } p_y\in f^{-1}(y), & \text{if } \partial f^{-1}(y)=\emptyset.\end{cases}$$

Let $X_0=X-L$, where $L=\cup\{L(y)\mid y\in Y\}$. Then X_0 is closed in X, and X_0 is a T_2 isocompact wM space. Let $h: X_0\to X$ be defined by $h(x)=x$ for each $x\in X_0$. Then $g=f\circ h$ is a perpect map of X_0 onto Y. Therefore Y is a T_2 isocompact (see [2]) and wM (see [4]) space. [Note that a space being a T_2 isocompact wM space is a perpect property.] Q.E.D.

References

[1] A. V. Arhangel'skii: On a class of spaces containing all metric spaces and all locally bicompact spaces. Soviet Math. Dokl., **4**, 1051–1055 (1963).

[2] P. Bacon: The compactness of countably compact spaces. Pacific J. Math., **32**(3), 587–592 (1970).

[3] R. E. Hodel: Metrizability of topological spaces. ibid., **55**(2), 441–459 (1974).

[4] T. Ishii: *wM* spaces and closed maps. Proc. Japan Acad., **46**, 16–21 (1970).

[5] E. Michael: A note on closed maps and compact sets. Israel J. Math., **2**, 173–176 (1964).

[6] J. Nagata: Quotient and biquotient spaces. ibid., **45**, 25–29 (1969).

[7] A. H. Stone: Metrizability of decomposition spaces. Proc. Amer. Math. Soc., **7**, 690–700 (1956).

[8] J. E. Vaughan: Spaces of countable and point countable type. Trans. Amer. Math. Soc., **51**, 341–350 (1970).

[9] H. H. Wicke: On the Hausdorff open continuous images of Hausdorff paracompact *p* spaces. ibid., **22**, 136–140 (1969).