94. On a Certain Condition for P-valently Starlikeness

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Summary. The object of the present paper is to derive a sufficient condition for p-valently starlike functions in the unit disk.

1. Introduction. Let $\mathcal{A}(p)$ be the class of functions of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n \qquad (p\in\mathcal{N}=\{1,2,3,\cdots\})$$

which are regular in the unit disk $\mathcal{D}=\{z:|z|<1\}$.

A function $f(z)$ belonging to the class $\mathcal{A}(p)$ is said to be p-valently starlike if and only if it satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \qquad (z\in\mathcal{D}).$$

We denote by $\mathcal{S}(p)$ the subclass of $\mathcal{A}(p)$ consisting of functions which are p-valently starlike in the unit disk $\mathcal{D}$. The class $\mathcal{S}(p)$ was introduced by Nunokawa [1], and was recently studied by Nunokawa and Owa [2].

A function $f(z)$ in the class $\mathcal{A}(p)$ is said to be p-valently typically real if and only if it has real values on the real axis and non-real values elsewhere.

2. Main theorem. We begin with the statement and the proof of our main result.

Theorem. *Let $f(z)$ be in the class $\mathcal{A}(p)$ and assume that*

$$(1) \qquad \left|\arg\left(\frac{f'(z)}{z^{p-1}}\right)\right|<\frac{\pi}{2}\alpha \qquad (z\in\mathcal{D})$$

and

$$(2) \qquad \left\{\operatorname{Im}\left(\frac{f'(z)}{z^{p-1}}\right)\right\}\{\operatorname{Im}(e^{-i\beta}z)\}\neq 0 \qquad (z\in\mathcal{D}(\beta))$$

for some real α $(0<\alpha\leqq 1)$ and β $(0\leqq\beta<\pi)$, where

$$\mathcal{D}(\beta)=\{z: 0<|z|<1 \text{ and } (\arg(z)-\beta)(\arg(z)-\beta-\pi)\neq 0\}.$$

Then we have

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\pi}{2}\alpha \qquad (z\in\mathcal{D}),$$

and therefore, $f(z)$ is p-valently starlike in the unit disk $\mathcal{D}$, or $f(z)\in\mathcal{S}(p)$.

Proof. Applying the same manner as in the proof by Ruscheweyh [3], we see that

$$\frac{f(z)}{zf'(z)}=\int_0^1\frac{f'(tz)}{f'(z)}dt$$
$$=\frac{z^{p-1}}{f'(z)}\int_0^1 t^{p-1}\frac{f'(tz)}{(tz)^{p-1}}dt \qquad \text{for } z\in\mathcal{D}.$$

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Therefore, it follows that

$$\arg\left\{t^{p-1}\frac{f'(tz)}{(tz)^{p-1}}\right\}=\arg\left\{\frac{f'(tz)}{(tz)^{p-1}}\right\}. \tag{3}$$

Using (1), (2), and (3), we see that if

$$0<\arg\left\{\frac{f'(z)}{z^{p-1}}\right\}<\frac{\pi}{2}\alpha,$$

then the integration

$$\int_0^1 t^{p-1}\frac{f'(tz)}{(tz)^{p-1}}\,dt$$

lies in the same convex sector $C=\{z: 0<\arg(z)<\pi\alpha/2\}$ and by the same reason as the above, if

$$0>\arg\left\{\frac{f'(z)}{z^{p-1}}\right\}>-\frac{\pi}{2}\alpha,$$

then

$$0>\arg\left\{\int_0^1 t^{p-1}\frac{f'(tz)}{(tz)^{p-1}}\,dt\right\}>-\frac{\pi}{2}\alpha.$$

Hence, we have

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\pi}{2}\alpha \qquad (z\in\mathcal{D}),$$

which implies that $f(z)$ is p-valently starlike in the unit disk $\mathcal{D}$. This completes the assertion of the theorem.

From the above theorem, we easily have the following corollaries.

Corollary 1. *Let $f(z)\in\mathcal{A}(p)$ and assume that*

$$\left|\arg\left(\frac{f'(z)}{z^{p-1}}\right)\right|<\frac{\pi}{2}\alpha \qquad (z\in\mathcal{D})$$

for some real α $(0<\alpha\leqq 1)$, and that $f'(z)/z^{p-1}$ is typically real in the unit disk $\mathcal{D}$. Then $f(z)$ belongs to the class $\mathcal{S}(p)$ and

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|<\frac{\pi}{2}\alpha \qquad (z\in\mathcal{D}).$$

Corollary 2. *Let $f(z)\in\mathcal{A}(p)$ and assume that*

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\}>0 \qquad (z\in\mathcal{D}),$$

and that $f'(z)/z^{p-1}$ is typically real in the unit disk $\mathcal{D}$. Then $f(z)$ belongs to the class $\mathcal{S}(p)$, or

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \qquad (z\in\mathcal{D}).$$

Corollary 3. *Let $f(z)\in\mathcal{A}(1)$ and assume that $f'(z)$ is typically real in the unit disk $\mathcal{D}$, and that*

$$|\arg(f'(z))|<\frac{\pi}{2}\alpha \qquad (z\in\mathcal{D}).$$

Then $f(z)$ is univalently starlike in $\mathcal{D}$ and

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|<\frac{\pi}{2}\alpha \qquad (z\in\mathcal{D}).$$

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References

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