11. *Existence of the Perturbed Solutions of Semilinear Elliptic Equation in the Singularly Perturbed Domains*

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In the previous paper [4], we have studied the asymptotic behaviors of the following semilinear elliptic equation defined on the singularly perturbed domain $\Omega(\zeta)$ with the Neumann boundary condition where $\Omega(\zeta) = D_1 \cup D_2 \cup Q(\zeta)$ and the moving portion $Q(\zeta)$ approaches a line segment L as $\zeta \to 0$.

$$\text{(1.1)} \qquad \begin{cases} \Delta v + f(v) = 0 & \text{in } \Omega(\zeta), \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega(\zeta), \end{cases}$$

where $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ is the Laplacian and ν is the unit normal vector on $\partial\Omega(\zeta)$ and f is a real valued smooth function on $\boldsymbol{R}$. We have proved in [4] that any solution v_ζ for small $\zeta > 0$, is approximated by some triple of solutions (w_1, w_2, V) of the following system of equations

$$\text{(1.2)} \qquad \begin{cases} \Delta w_i + f(w_i) = 0 & \text{in } D_i, \\ \partial w_i / \partial \nu = 0 & \text{on } \partial D_i, \end{cases} \quad (i=1,2)$$

$$\text{(1.3)} \qquad \begin{cases} d^2V/dz^2 + f(V) = 0 & z \in L, \\ V|_{\partial D_i \cap \partial L} = w_i|_{\partial D_i \cap \partial L} & (i=1,2), \end{cases}$$

where z is an adequate variable along L. In view of the above characterization of the solutions (1.1) the following inverse problem naturally arise, i.e. for any given triple of solutions $\{w_1, w_2, V\}$ of the system of the equations (1.2) and (1.3), is there a family of functions $\{v_\zeta\}_{0<\zeta<\zeta_*}$ such that each $v_\zeta \in C^\infty(\overline{\Omega(\zeta)})$ is a solution of (1.1) and satisfies the following asymptotic conditions,

$$v_\zeta \sim w_i \text{ in } D_i \ (i=1,2), \qquad v_\zeta \sim V \text{ in } Q(\zeta)$$

for small $\zeta > 0$ in some sense.

In this paper we report an affirmative answer to the above problem under some non-degeneracy condition of $\{w_1, w_2, V\}$.

First we establish the situation. We set the domain $\Omega(\zeta)$ in the following form:

$$\Omega(\zeta) = D_1 \cup D_2 \cup Q(\zeta)$$

where D_i $(i=1,2)$ and $Q(\zeta)$ are defined in the following conditions (A.1) and (A.2) where $x' = (x_2, x_3, \cdots, x_n) \in \boldsymbol{R}^{n-1}$.

(A.1) D_1 and D_2 are bounded domains in $\boldsymbol{R}^n$ where $\bar{D}_1 \cap \bar{D}_2 = \emptyset$ and each D_i has a smooth boundary ∂D_i and the following conditions hold for some positive constant $\zeta_* > 0$.

$$\bar{D}_1 \cap \{x=(x_1, x') \in \boldsymbol{R}^n \mid x_1 \leqq 1, |x'|<3\zeta_*\} = \{(1, x') \in \boldsymbol{R}^n \mid |x'|<3\zeta_*\}$$

$$\bar{D}_2 \cap \{x=(x_1, x') \in \boldsymbol{R}^n \mid x_1 \geqq -1, |x'|<3\zeta_*\} = \{(-1, x') \in \boldsymbol{R}^n \mid |x'|<3\zeta_*\}$$

(A.2) $$Q(\zeta)=R_1(\zeta) \cup R_2(\zeta) \cup \Gamma(\zeta)$$

$$R_1(\zeta)=\{(x_1, x') \in \boldsymbol{R}^n \mid 1-2\zeta<x_1 \leqq 1, |x'|<\zeta\rho((x_1-1)/\zeta)\}$$

$$R_2(\zeta)=\{(x_1, x') \in \boldsymbol{R}^n \mid -1 \leqq x_1<-1+2\zeta, |x'|<\zeta\rho((-1-x_1)/\zeta)\}$$

$$\Gamma(\zeta)=\{(x_1, x') \in \boldsymbol{R}^n \mid -1+2\zeta \leqq x_1 \leqq 1-2\zeta, |x'|<\zeta\}$$

where $\rho \in C^0((-2,0]) \cap C^\infty((-2,0))$ is a positive function such that $\rho(0)=2$, $\rho(s)=1$ for $s \in (-2,-1)$, $d\rho/ds>0$ for $s \in (-1,0)$ and the inverse function $\rho^{-1}: (1,2) \to (-1,0)$ satisfies $\lim_{\xi \uparrow 2-0}(d^k\rho^{-1}/d\xi^k)=0$ holds for any positive integer $k \geqq 1$. We put

$$p_1=(1,0,\cdots,0), \qquad p_2=(-1,0,\cdots,0),$$
$$L=\{(z,0,\cdots,0) \in \boldsymbol{R}^n \mid -1<z<1\}.$$

We impose the following conditions.

(A.3) $$f \in C^\infty(\boldsymbol{R}), \quad \limsup_{\xi \to +\infty} f(\xi)<0, \quad \liminf_{\xi \to -\infty} f(\xi)>0.$$

(A.4) There exists a system of solutions $\{w_1, w_2, V\}$ in $C^\infty(\bar{D}_1) \times C^\infty(\bar{D}_2) \times C^\infty([-1,1])$ of (1.2) and (1.3).

Definition. For the above solutions $\{w_1, w_2, V\}$ in (A.4), we denote by $\{w_k\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$, respectively, the system of the eigenvalues arranged in increasing order (counting multiplicity) of the following eigenvalue problems (1.4) and (1.5),

(1.4) $$\begin{cases} \Delta\phi+f'(w)\phi+\omega\phi=0 & \text{in } D_1 \cup D_2, \\ \partial\phi/\partial\nu=0 & \text{on } \partial D_1 \cup \partial D_2, \end{cases}$$

where

$$w(x)=\begin{cases} w_1(x) & \text{for } x \in D_1, \\ w_2(x) & \text{for } x \in D_2, \end{cases}$$

(1.5) $$\begin{cases} \dfrac{d^2S}{dz^2}+f'(V)S+\lambda S=0 & -1<z<1, \\ S(1)=S(-1)=0. \end{cases}$$

We assume the following non-degeneracy condition of $\{w_1, w_2, V\}$.

(A.5) $$\{\omega_k\}_{k=1}^\infty \cap \{\lambda_k\}_{k=1}^\infty=\varnothing, \qquad \{\omega_k\}_{k=1}^\infty \cup \{\lambda_k\}_{k=1}^\infty \not\ni 0.$$

Theorem. *Assume $n \geqq 3$ and the assumptions* (A.1)–(A.5). *Then, for any $\zeta \in (0, \zeta_*)$, there exists a solution v_ζ of* (1.1) *such that*

(1.6) $$\lim_{\zeta \to 0} \sup_{x \in D_1 \cup D_2} |v_\zeta(x)-w(x)|=0,$$

(1.7) $$\lim_{\zeta \to 0} \sup_{x \in Q(\zeta)} |v_\zeta(x_1, x')-V(x_1)|=0.$$

Sketch of proof. In the proof of Theorem, the results and methods obtained in [4] and [5] are essentially applied, especially in our delicate reduction of (1.1) to the problem of finite dimension. By these methods, we can construct an approximate solution $A_\zeta \in C^\infty(\overline{\Omega(\zeta)})$ such that

(2.1) $$\begin{cases} \lim_{\zeta \to 0} \sup_{x \in D_1 \cup D_2} |A_\zeta(x)-w(x)|=0 \\ \lim_{\zeta \to 0} \sup_{x \in Q(\zeta)} |A_\zeta(x_1, x')-V(x_1)|=0 \end{cases}$$

$$\text{(2.2)}\qquad \begin{cases} \lim\limits_{\zeta\to 0} \sup\limits_{x\in\Omega(\zeta)} |\Delta A_\zeta(x)+f(A_\zeta(x))|=0 \\ \partial A_\zeta(x)/\partial\nu=0 \quad \text{on} \quad \partial\Omega(\zeta). \end{cases}$$

We project (1.1) to the subspace of $H^1(\Omega(\zeta))$ by using the eigenfunctions of the linearized problem at A_ζ.

Let $\{\mu_k(\zeta)\}_{k=1}^\infty$ and $\{\Phi_{k,\zeta}\}_{k=1}^\infty$ be, respectively, the eigenvalues (counting multiplicity) arranged in increasing order and the complete system of orthonormalized eigenfunctions in $L^2(\Omega(\zeta))$. By [5], we have the following decompositions

$$\text{(2.3)}\qquad \{\mu_k(\zeta)\}_{k=1}^\infty=\{\omega_k(\zeta)\}_{k=1}^\infty \cup \{\lambda_k(\zeta)\}_{k=1}^\infty$$

$$\text{(2.4)}\qquad \{\Phi_{k,\zeta}\}_{k=1}^\infty=\{\phi_{k,\zeta}\}_{k=1}^\infty \cup \{\psi_{k,\zeta}\}_{k=1}^\infty$$

where

$$\lim_{\zeta\to 0}\omega_k(\zeta)=\omega_k, \quad \lim_{\zeta\to 0}\lambda_k(\zeta)=\lambda_k \quad (k\geqq 1)$$

and

$$\text{(2.5)}\qquad \begin{cases} \overline{\lim\limits_{\zeta\to 0}} \|\phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}<+\infty \\ \lim\limits_{\zeta\to 0} \|\psi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}=+\infty \end{cases} \quad (k\geqq 1)$$

$$\text{(2.6)}\qquad \|\psi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}\sim O(\zeta^{-(n-1)/2}) \qquad (k\geqq 1),$$

$$\text{(2.7)}\qquad \|\psi_{k,\zeta}\|_{L^1(\Omega(\zeta))}\sim O(\zeta^{(n-1)/2}) \qquad (k\geqq 1).$$

Let

$$X(\zeta)=H^1(\Omega(\zeta)), \qquad X_1(\zeta)=\text{L.h.}[\{\phi_{k,\zeta}\}_{k=1}^q \cup \{\psi_{k,\zeta}\}_{k=1}^q]$$

and

$$X_2(\zeta)=\overline{\text{L.h.}[\{\phi_{k,\zeta}\}_{k=q+1}^\infty \cup \{\psi_{k,\zeta}\}_{k=q+1}^\infty]}^{\text{in } X(\zeta)}$$

where q is a adequately fixed large natural number determined by f. We seek the solution in the form

$$v(x)=A_\zeta(x)+\Phi_\zeta^{(1)}+\Phi_\zeta^{(2)} \quad \text{where} \quad \Phi_\zeta^{(i)}\in X_i(\zeta) \quad (i=1,2).$$

Project (1.1) to the subspaces $X_1(\zeta)$ and $X_2(\zeta)$ by the following operator P_ζ on $L^2(\Omega(\zeta))$,

$$P_\zeta\Phi(x)=\sum_{k=1}^q ((\Phi\cdot\phi_{k,\zeta})_{L^2(\Omega(\zeta))}\phi_{k,\zeta}(x)+(\Phi\cdot\psi_{k,\zeta})_{L^2(\Omega(\zeta))}\psi_{k,\zeta}(x)).$$

The difficulty of the reduction is due to the existence of the singularly behaving eigenfunctions $\{\psi_{k,\zeta}\}_{k=1}^\infty$ (cf. (2.5)) which are associated with the partial collapse of $\Omega(\zeta)$. By the elaborate estimate (2.6) and (2.7), the operator P_ζ maps $L^\infty(\Omega(\zeta))$ into $L^\infty(\Omega(\zeta))$ and its operator norm is bounded in $\zeta>0$. Thus we can carry a good formulation in $L^\infty(\Omega(\zeta))$, i.e. we can obtain the finite dimensional equation with respect to the variable $\tau=(\tau_1,\tau_2,\cdots,\tau_{2q})$ by putting

$$\Phi_{\tau,\zeta}^{(1)}(x)=\sum_{k=1}^q (\tau_k\phi_{k,\zeta}(x)+\tau_{q+k}\bar{\psi}_{k,\zeta}(x))$$

where $\bar{\psi}_{k,\zeta}(x)=\psi_{k,\zeta}(x)/\|\psi_{k,\zeta}\|_{L^2(\Omega(\zeta))}$.

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