93. *On the Darboux Transformation of Second Order Ordinary Differential Operator*

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1. Introduction. The main purpose of the present paper is to study the relation between the *Darboux transformation* of 2-nd order ordinary differential operator and the recursion formula called the *Lenard relation.* The Darboux transformation is studied in [1], [3], [4], [5] and [6] for the 1-dimensional Schrödinger operator and the ordinary differential operator of Fuchsian type. In this paper we supplement them with the description of more general aspect of the theory.

2. Darboux transformation. Consider the 2-nd order ordinary differential operator $L(u)=\partial^2-u(x)$, $\partial=d/dx$, where $u(x)$ is a complex analytic function defined in a region $\Omega\subset \boldsymbol{P}_1$. Suppose that $u(x)$ is holomorphic at $x=a\in\Omega$ and let $y_j(x)$ $(j=1,2)$ be the fundamental system of solutions of the differential equation

$$L(u)y=y''-u(x)y=0, \qquad '=d/dx$$

such that $W(y_1(a), y_2(a))=E$, where $W(f,g)=\begin{pmatrix} f & g \\ f' & g'\end{pmatrix}$ is the Wronskian matrix and E is the unit matrix of size 2. For $\zeta=[\xi_1:\xi_2]\in\boldsymbol{P}_1$, put $v(x,\zeta)=(\partial/\partial x)\log(\xi_1y_1(x)+\xi_2y_2(x))$ and $A_\pm(\zeta)=\partial\pm v(x,\zeta)$. Then the factorization $L(u)=A_+(\zeta)A_-(\zeta)$ follows. On the other hand, put

$$L^*(u;\zeta)=A_-(\zeta)A_+(\zeta).$$

$L^*(u;\zeta)$ is the 2-nd order ordinary differential operator parametrized by $\zeta\in\boldsymbol{P}_1$. We call $L^*(u;\zeta)$ the Darboux transformation of $L(u)$. Put $u^*(x,\zeta)=u(x)-2(\partial/\partial x)v(x,\zeta)$, which is analytic in $\Omega^*=\Omega\setminus\{\text{zeros of } \sum_{j=1}^{2}\xi_jy_j(x)\}$, then $L^*(u;\zeta)=\partial^2-u^*(x,\zeta)$ follows.

3. Lenard relation. Define the function $Q_n(x)$ $(n=1,2,\cdots)$ by the recursion formula

$$2Q'_{n+1}(x)=u'(x)Q_n(x)+2u(x)Q'_n(x)-2^{-1}Q'''_n(x)$$

with $Q_0(x)=1$. It is known that $Q_n(x)$ are polynomials of $u, u', \cdots, u^{(2n-2)}$ with constant coefficients (cf. [7]). Of course, while an arbitrary constant appears when we integrate $Q'_n(x)$ to obtain $Q_n(x)$ itself, we can define uniquely $Q_n(x)$ by putting them zero. Hence we can define the nonlinear differential operators $Z_n(u)$ and $X_n(u)$ by $Z_n(u(x))=2Q_n(x)$ and $X_n(u(x))=2Q'_n(x)=\partial Z_n(u(x))$. Then we can rewrite the above recursion formula as

$$X_n(u)=(2^{-1}u'+u\partial-4^{-1}\partial^3)Z_{n-1}(u),$$

which is called the Lenard relation. $Z_n(u)$ turns out to be the $(2n-2)$-th order differential polynomial. For example, we have

$$Z_0(u)=2,\quad Z_1(u)=u,\quad Z_2(u)=4^{-1}(3u^2-u''),$$
$$X_0(u)=0,\quad X_1(u)=u',\quad X_2(u)=4^{-1}(6uu'-u''').$$

4. Main results. Put $B_\pm(\zeta)=\pm\partial-(\pm v_x(x,\zeta)/v(x,\zeta)-2v(x,\zeta))$ and $C_\pm(\zeta)=\pm\partial+2v(x,\zeta)$. Then we have

Theorem 1. *The equalities*

(1) $$B_+(\zeta)X_n(u^*(x;\zeta))=B_-(\zeta)X_n(u(x))$$

and

(2) $$C_+(\zeta)Z_n(u^*(x;\zeta))=C_-(\zeta)Z_n(u(x))$$

are valid for all $n\in\boldsymbol{N}$.

While the proof of this theorem, which is given in [6], is elementary, the formula (1) and (2) yield many interesting results in the transformation theory of the higher order KdV equations.

5. Application. For $\tau_\mu\in\boldsymbol{C}$ $(\mu=0,\infty)$, define the solutions $f_\mu(x,\tau_\mu)$ $(\mu=0,\infty)$ of the equation $L(u)y=0$ by $f_0(x,\tau_0)=y_1(x)+\tau_0 y_2(x)$ and $f_\infty(x,\tau_\infty)=\tau_\infty y_1(x)+y_2(x)$ and put

$$u_\mu^*=u_\mu^*(x,\tau_\mu)=u(x)-2(\partial/\partial x)^2\log f_\mu(x,\tau_\mu),\qquad \mu=0,\infty.$$

Then we have the following (cf. [6]).

Theorem 2. *Let $u(x)$ be the solution of the $(2n-2)$-th order algebraic differential equation*

$$Z_n(u)+\sum_{\nu=0}^{n-1}c_\nu Z_\nu(u)=0,\qquad c_\nu\in\boldsymbol{C},\ \nu=0,1,\cdots,n-1.$$

Then the functions

$$-2^{-1}f_\mu(x,\tau_\mu)^2\left\{Z_n(u_\mu^*)+\sum_{\nu=0}^{n-1}c_\nu Z_\nu(u_\mu^*)\right\}$$

are independent of x and analytic in τ_μ; we denote it by $\phi_\mu(\tau_\mu)$ $(\mu=0,\infty)$ respectively. Moreover u_μ^* solve the n-th KdV equation

$$\pm\phi_\mu\partial u_\mu^*/\partial\tau_\mu-X_n(u_\mu^*)-\sum_{\nu=1}^{n-1}c_\nu X_\nu(u_\mu^*)=0$$

and the $(2n+1)$-th order algebraic differential equation

$$X_{n+1}(u_\mu^*)+\sum_{\nu=1}^{n-1}c_\nu X_{\nu+1}(u_\mu^*)=0.$$

On the other hand, suppose that the coefficient $u=u(x,t)$ of $L(u)$ depends analytically also on the another complex parameter $t\in D\subset\boldsymbol{P}_1$ and solves the n-th KdV equation

$$\partial u/\partial t-X_n(u)-\sum_{\nu=1}^{n-1}c_\nu X_\nu(u)=0,\qquad c_\nu\in\boldsymbol{C},\ \nu=1,\cdots,n-1.$$

Assume that $u(x,t)$ is holomorphic at $x=a$ for all $t\in D$ and let $y_j(x,t)$ $(j=1,2)$ be the fundamental system of solutions of $L(u(x,t))y=0$ such that $W(y_1(a,t),y_2(a,t))=E$. Put

$$u^*(x,t;\zeta)=u(x,t)-2(\partial/\partial x)^2\log(\xi_1 y_1(x,t)+\xi_2 y(x,t))$$

for $\zeta=[\xi_1:\xi_2]\in\boldsymbol{P}_1$, then, by direct calculation, we have

Theorem 3. *The function $u^*=u^*(x,t;\zeta)$ solves the equation*

$$B_+(\zeta)\left(\partial u^*/\partial t-X_n(u^*)-\sum_{\nu=1}^{n-1}c_\nu X_\nu(u^*)\right)=0.$$

6. Examples. The method of Darboux transformation has been used to construct the exact solutions of the n-th KdV equation by many authors; see e.g. [3] for the multi soliton solutions and see [1], [4] and [6] for the rational solutions. Here we investigate the elliptic solutions. Let

$$\mathscr{P}(x)=x^{-2}+\sum_{\omega\neq 0}(x-\omega)^{-2}-\omega^{-2},\quad \omega=m\omega_1+n\omega_2,\quad m,n\in \boldsymbol{Z}$$

be the Weierstrass $\mathscr{P}$ function with the period ω_1, ω_2. Put $u=u(x;\alpha,\beta)=2\alpha^2\mathscr{P}(\alpha x)+\beta$. $\mathscr{P}(x)$ solves the algebraic differential equation $\mathscr{P}''-6\mathscr{P}^2+g=0$, where $g=30\sum_{\omega\neq 0}\omega^{-4}$ (see e.g. [2]). Hence, by direct calculation, one verifies

$$Z_2(u)+c_1Z_1(u)+c_2Z_0(u)=0,$$

where $c_1=-3\beta/2$ and $c_2=(3\beta^2-2\alpha^2 g)/8$. Let $\lambda_j(x;\alpha,\beta)$ $(j=1,2)$ be the fundamental system of solutions of the equation

$$\lambda''-u(x;\alpha,\beta)\lambda=0 \tag{3}$$

such that $W(\lambda_1(a;\alpha,\beta),\lambda_2(a;\alpha,\beta))=E$ for some $a\notin\{m\omega_1+n\omega_2\,|\,m,n\in\boldsymbol{Z}\}$ and put $f_0(x,\tau_0;\alpha,\beta)=\lambda_1(x;\alpha,\beta)+\tau_0\lambda_2(x;\alpha,\beta)$ and $f_\infty(x,\tau_\infty;\alpha,\beta)=\tau_\infty\lambda_1(x;\alpha,\beta)+\lambda_2(x;\alpha,\beta)$. Then, by Theorem 2,

$$u_\mu^*=u_\mu^*(x,\tau_\mu;\alpha,\beta)=u(x;\alpha,\beta)-2(\partial/\partial x)^2\log f_\mu(x,\tau_\mu;\alpha,\beta),\qquad \mu=0,\infty$$

turn out to solve the KdV equation

$$\pm\phi_\mu(\tau_\mu)\partial u_\mu^*/\partial\tau_\mu-X_2(u_\mu^*)-c_1X_1(u_\mu^*)=0,\qquad \mu=0,\infty$$

and the 5-th order algebraic differential equation

$$X_3(u_\mu^*)+c_1X_2(u_\mu^*)=0,\qquad \mu=0,\infty.$$

Let $\Lambda(x)$ be the non-trivial solution of the Lamé's differential equation

$$\Lambda''-(2\mathscr{P}(x)+\alpha^{-2}\beta)\Lambda=0,$$

then $\lambda(x)=\Lambda(\alpha x)$ solves (3). Hence $u_\mu^*(x,\tau_\mu;\alpha,\beta)$ are described by Lamé function if α and β are appropriately choosed. Moreover let

$$\sigma(x)=x\prod_{\omega\neq 0}(1-\omega^{-1}x)\exp(\omega^{-1}x+2^{-1}\omega^{-2}x^2)$$

be the Weierstrass sigma function. Since $\mathscr{P}(x)=(\partial/\partial x)^2\log\sigma(x)^{-1}$ is valid (cf. e.g. [2]),

$$u(x;\alpha,\beta)=2(\partial/\partial x)^2\log\theta(x;\alpha,\beta)$$

follows, where $\theta(x;\alpha,\beta)=\sigma(\alpha x)^{-1}\exp(4^{-1}\beta x^2)$. Hence if we put

$$\theta_\mu^*(x,\tau_\mu;\alpha,\beta)=\theta(x;\alpha,\beta)/f_\mu(x,\tau_\mu;\alpha,\beta) \tag{4}$$

then we have

$$u_\mu^*(x,\tau_\mu;\alpha,\beta)=2(\partial/\partial x)^2\log\theta_\mu^*(x,\tau_\mu;\alpha,\beta).$$

Thus, if we employ the τ-functions θ and θ_μ^* then the Darboux transformation can be represented as the division by the solution $f_\mu(x,\tau_\mu;\alpha,\beta)$ by (4).

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