87. The Period Map of a 4-parameter Family of K3 Surfaces and the Aomoto-Gel'fand Hypergeometric Function of Type (3, 6)[†]

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We show that one of Aomoto-Gel'fand hypergeometric functions ([4]) can be interpreted as the period map of a 4-dimensional family of $K3$ surfaces, of which the target is the 4-dimensional Hermitian symmetric bounded domain of type IV. The corresponding system of differential equations has six linearly independent solutions which are quadratically related. (Such systems are recently studied in [10].) This fact confers an algebro-geometric decoration to the Aomoto-Gel'fand functions as the relation between the elliptic modular function and the corresponding equation does to the Gauss hypergeometric function. Details will be given in [6].

We describe a family of $K3$ surfaces. Let

$$l_j=\{(t^1, t^2, t^3)\in \boldsymbol{CP}^2 \mid v_{1j}t^1+v_{2j}t^2+v_{3j}t^3=0\} \qquad (0\leqq j\leqq 6)$$

be six lines in general position in the complex projective plane $\boldsymbol{CP}^2$ with homogeneous coordinates (t^1, t^2, t^3) and let $S(l)$ be the minimal smooth model of the two-fold cover $S'(l)$ of $\boldsymbol{CP}^2$ branching along the line configuration $l=\{l_1, \cdots, l_6\}$. For a fixed l, the surface $S(l)$ is a $K3$ surface, i.e., there is a unique holomorphic 2-form

$$\eta(l)=\prod_{j=1}^{6}(v_{1j}s^1+v_{2j}s^2+v_{3j})^{-1/2}ds^1\wedge ds^2, \tag{1}$$

up to constant multiplication, and the rank of the second homology group $H_2(S(l), \boldsymbol{Z})$ is 22. In this case, there are 16 linearly independent cycles; 15 exceptional curves coming from the 15 double points of $S'(l)$ and a section when considered $S(l)$ as an elliptic surface over $\boldsymbol{CP}^1$. We can take a system $\gamma_1'(l), \cdots, \gamma_6'(l)\in H_2(S(l)\boldsymbol{Z})$ of six (transcendental) cycles orthogonal to the algebraic cycles such that there exists another system $\gamma_1(l), \cdots, \gamma_6(l)\in H_2(S(l), \boldsymbol{Z})$ which is dual to γ_j' $(1\leqq j\leqq 6)$, i.e., $\gamma_i'\cdot\gamma_j=\delta_{ij}$ (Kronocker's symbol) and that its intersection matrix $(\gamma_i'\cdot\gamma_j')$ $(1\leqq i, j\leqq 6)$ takes the fixed form $I=(I_{ij})$, which is symmetric, integral and with the signature $(2+, 4-)$. The vector $\omega(l)=(\omega_1(l), \cdots, \omega_6(l))$, where $\omega_j(l)=\int_{\gamma_j(l)}\eta(l)$ $(1\leqq j\leqq 6)$, is called the period of $S(l)$ and it satisfies the Riemann relation and the Riemann inequality as follows

$$\sum_{i,j} I_{ij}\omega_i(l)\omega_j(l)=0 \tag{2}$$

$$\sum_{i,j} I_{ij}\omega_i(l)\bar{\omega}_j(l)>0. \tag{3}$$

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Now we let l vary in the space M of configurations of six projective lines and let the cycles $\gamma_j(l)$ depend continuously on l. Then the correspondence sending l to the ratio $\omega_1(l):\cdots:\omega_6(l)$ gives a multi-valued map $\varphi: M\to Q\subset \boldsymbol{C}P^5$ where $Q=\{(z^1,\cdots,z^6)\in \boldsymbol{C}P^6 \mid \sum_{i,j} I_{ij}z^iz^j=0\}$. The multi-valuedness of φ is expressed by a subgroup of $\Gamma=\{X\in GL(6,\boldsymbol{Z}) \mid {}^tXIX=I\}/\{\pm 1\}$. If we express elements of the space M by a 3 by 6 matrix (v_{ij}) $(1\leqq i\leqq 3, 1\leqq j\leqq 6)$ then the map φ is invariant under the action of $SL(3,\boldsymbol{C})$ on the left and the action $H=(\boldsymbol{C}^\times)^6$ from the right. Therefore φ is defined on the 4-dimensional factor space $X=SL(3,6)\backslash M/H$. Let us choose, for example, a system of local coordinates $(x^1,\cdots,x^4)$ of X as follows:

$$(v_{ij})=\begin{pmatrix}1 & 0 & 0 & 1 & 1 & 1\\ 0 & 1 & 0 & 1 & x^1 & x^2\\ 0 & 0 & 1 & 1 & x^3 & x^4\end{pmatrix}$$

i.e.

$$(4)\qquad l_1=\{t^1=0\},\quad l_2=\{t^2=0\},\quad l_3=\{t^3=0\},\quad l_4=\{t^1+t^2+t^3=0\},$$
$$l_5=\{t^1+x^1t^2+x^3t^3=0\},\qquad l_6=\{t^1+x^2t^2+x^4t^3=0\}.$$

Then by the theory developed in [10], there is a system of linear differential equations

$$(5)\qquad \frac{\partial^2 u}{\partial x^i\partial x^j}=g_{ij}\frac{\partial^2 u}{\partial x^1\partial x^4}+\sum_{k=1}^{3} a_{ij}^k\frac{\partial u}{\partial x^k}+a_{ij}^0 u,\qquad 1\leqq i,j\leqq 6$$

of rank (= dimension of the solution space) six such that the ratio of a system of linearly independent solution (called a projective solution) is exactly φ, that the quadratic form $g=\sum_{i,j} g_{ij}dx^idx^j$ is conformal to the pull back of the canonical *flat* conformal structure on $Q\subset \boldsymbol{C}P^5$ and that other coefficients a_{ij}^k and a_{ij}^0 $(1\leqq i,j,k\leqq 6)$ are determined by g.

Here we recall the framework of the Aomoto-Gel'fand hypergeometric differential equation associated with the Grassmannian $G_{3,6}$ and see that our system (5) is its special case. Let $M(k,n)$ be the set of k by n matrices $v=(v_{ij})$ and consider the integral

$$(6)\qquad \Phi(v)=\int_\Delta \prod_{j=1}^n \left(\sum_{i=1}^k v_{ij}t^i\right)^{\alpha_j-1} dt$$

where Δ is a region of the $(k-1)$-dimensional sphere $S^{k-1}\subset \boldsymbol{R}^k$ and dt is the induced measure of the standard measure of the Euclidean space $\boldsymbol{R}^k$ onto S^{k-1} and $\sum_{j=1}^n \alpha_j=n-k$. The function $\Phi(v)$ is invariant under the action of $SL(k,\boldsymbol{C})$ from the left and the action of $H_n=(\boldsymbol{C}^\times)^n$ from the right so that it satisfies

$$(7)\qquad \sum_{i=1}^k v_{ij}\frac{\partial}{\partial v_{ij}}\Phi=(\alpha_j-1)\Phi\qquad (H_n\text{-invariance})$$

for $1\leqq j\leqq n$ and

$$(8)\qquad \sum_{j=1}^n v_{ij}\frac{\partial}{\partial v_{kj}}\Phi=-\delta_i^k\Phi\qquad (SL(k,\boldsymbol{C})\text{-invariance})$$

for $1\leqq i,k\leqq k$. Important equalities are

$$(9)\qquad \frac{\partial^2}{\partial v_{ip}\partial v_{jq}}\Phi=\frac{\partial^2}{\partial v_{iq}\partial v_{jp}}\Phi$$

for $1\leqq i, j\leqq k$, $1\leqq p, q\leqq n$. The system of linear differential equations (7), (8) and (9) is called *the Aomoto-Gel'fand hypergeometric equation of type* (k, n) ([1] and [3]) and denoted by $E(k, n; \alpha_1, \cdots, \alpha_n)$. It is a holonomic system, that is, its rank is finite. The systems $E(2, 4; \alpha_1, \cdots, \alpha_4)$ and $E(2, n; \alpha_1, \cdots, \alpha_n)$ $(n\geqq 5)$ can be naturally considered, respectively, to be the Gauss hypergeometric equation and the Appell-Lauricella hypergeometric system F_D in $n-3$ variables.

The system $E(3, 6; \alpha_1, \cdots, \alpha_6)$ reduces to the following system with unknown u if one uses the independent variables $x^1, \cdots, x^4$ appeared in the normalization (4).

$$(\alpha_2+\alpha_3+\alpha_4-1+\theta_1+\theta_2+\theta_3+\theta_4)\theta_1 u=x^1(\theta_1+\theta_3+1-\alpha_5)(\theta_1+\theta_2+\alpha_2)u$$
$$(\alpha_2+\alpha_3+\alpha_4-1+\theta_1+\theta_2+\theta_3+\theta_4)\theta_2 u=x^2(\theta_2+\theta_4+1-\alpha_6)(\theta_1+\theta_2+\alpha_2)u$$
$$(\alpha_2+\alpha_3+\alpha_4-1+\theta_1+\theta_2+\theta_3+\theta_4)\theta_3 u=x^3(\theta_1+\theta_3+1-\alpha_5)(\theta_3+\theta_4+\alpha_3)u$$
$$(\alpha_2+\alpha_3+\alpha_4-1+\theta_1+\theta_2+\theta_3+\theta_4)\theta_4 u=x^4(\theta_2+\theta_4+1-\alpha_6)(\theta_3+\theta_4+\alpha_3)u$$

where $\theta_i=x^i\partial/\partial x^i$. This system can be written in the form of (5) as follows:

$$(10)_\alpha \qquad \frac{\partial^2 u}{\partial x^i\partial x^j}=G_{ij}\frac{\partial^2 u}{\partial x^1\partial x^4}+\sum_{k=1}^{3}A_{ij}^k(\alpha)\frac{\partial u}{\partial x^k}+A_{ij}^0(\alpha)u,$$

$1\leqq i, j\leqq 6$. The coefficients $G_{ij}=G_{ji}$'s of the principal part are independent of the α_j's and are given as follows:

$$G_{12}=\frac{x^4-x^3}{x^1-x^2},\quad G_{13}=\frac{x^4-x^2}{x^1-x^3},\quad G_{24}=\frac{x^3-x^1}{x^2-x^4},\quad G_{34}=\frac{x^2-x^1}{x^3-x^4},$$
$$G_{11}=\frac{x^2x^3-x^4}{x^1(1-x^1)}-\frac{x^3(x^4-x^2)}{x^1(x^1-x^3)}-\frac{x^2(x^4-x^3)}{x^1(x^1-x^2)},$$
$$G_{22}=\frac{x^1x^4-x^3}{x^2(1-x^2)}-\frac{x^1(x^3-x^4)}{x^2(x^2-x^1)}-\frac{x^4(x^3-x^1)}{x^2(x^2-x^4)},$$
$$G_{33}=\frac{x^1x^4-x^2}{x^3(1-x^3)}-\frac{x^1(x^2-x^4)}{x^3(x^3-x^1)}-\frac{x^4(x^2-x^1)}{x^3(x^3-x^4)},$$
$$G_{44}=\frac{x^2x^3-x^1}{x^4(1-x^4)}-\frac{x^3(x^1-x^2)}{x^4(x^4-x^3)}-\frac{x^2(x^1-x^3)}{x^4(x^4-x^2)},\qquad G_{14}=G_{23}=1.$$

On the other hand, it is clear from the representations (1) and (6) that our system (5) is equivalent to the system $(10)_\alpha$ with $\alpha_j=1/2$ for $1\leqq j\leqq 6$. Therefore we know that the coefficients g_{ij} of (5) are equal to G_{ij}, so that the quadratic form $g=\sum_{i,j}G_{ij}dx^i dx^j$ is *conformally flat*.

The algebro-geometric interpretation of the system $E(3, 6; 1/2, \cdots, 1/2)$ given above shows that the system $E(3, 6; \alpha_1, \cdots, \alpha_6)$, out of many systems $E(k, n; \alpha_1, \cdots, \alpha_n)$ $(k\geqq 3)$, is a valuable analogy of the Gauss hypergeometric equation that has the fruitful relation with the elliptic modular function. To conclude this paper, we list up the correspondence between our situation of $K3$ surfaces and the situation of elliptic curves: (a) Configuration l of six lines in $CP^2\leftrightarrow$System of four points $p=(p_1, \cdots, p_4)$ on CP^1. (b) $K3$ surface $S(l)\leftrightarrow$Elliptic curve $E(p)$ obtained by the two fold cover of CP^1 branching at p. (c) Holomorphic 2-form $\eta(l)$ on $S(l)\leftrightarrow$Holomorphic 1-form $\eta(p)$ on $E(p)$. (d) Transcendental cycles $\gamma_i'(l)\in H_2(S(l), Z)\leftrightarrow$Standard basis

$\gamma_1(p)$ and $\gamma_2(p)$ of $H_1(E(p), Z)$. (e) Intersection form I of the $\gamma_i'(l)$'s $\leftrightarrow$ Intersection form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $\gamma_1(p)$ and $\gamma_2(p)$. (f) Periods $\omega_i(l)=\int_{\gamma_i(l)} \eta_i(l) \leftrightarrow$ Periods $\omega_i(p)=\int_{\gamma_i(p)} \eta_i(p)$. (g) The Riemann inequality (2) $\leftrightarrow \operatorname{Im} \omega_1(p)/\omega_2(p) > 0$. (h) Period map $\varphi: l \to \omega(l) \in Q \leftrightarrow$ Period map $p \to \omega_1(p)/\omega_2(p) \in H=\{z \in C \mid \operatorname{Im} z>0\}$. (i) Group $\Gamma \leftrightarrow$ Group $PSL(2, Z)$. (j) System $E(3, 6; 1/2, \cdots, 1/2)$ $\leftrightarrow$ System $E(2, 4; 1/2, \cdots, 1/2)$. (k) System (5) under the normalization (4) $\leftrightarrow$ The hypergeometric differential equation $x(1-x)u''+(1-2x)u'-(1/4)u=0$ under the normalization $p_1=0$, $p_1=\infty$, $p_3=1$, $p_4=x$.

Note. Such correspondence can be also found using various families of curves of higher genera in place of $K3$ surfaces, which are studied in [2], [5], [10] and [11].

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