83. A Note on E-direct and S-inverse Systems

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Introduction. Let R be a fixed (not necessarily commutative) ring with unity. Throughout this paper we are concerned with left R-modules and M stands for a unitary R-module. The submodule generated by an element $a \in M$ is denoted by Ra. Like in Fleury [1], Goldie [2], Reddy and Satyanarayana [3], Satyanarayana [4] and Sharpe and Vamoes [5] we shall use the following terminology. A non-zero submodule K of M is called *essential* in M (or M is an *essential extension* of K) if $K \cap A = 0$ for any other submodule A of M, implies $A=0$. M has *finite Goldie dimension* (abbr. FGD) if M does not contain a direct sum of infinite number of non-zero submodules. Equivalently, M has FGD if for any strictly increasing sequence $H_0, H_1, \cdots$ of submodules of M, there is an integer i such that for every $k \geqslant i$, H_k is an essential submodule in H_{k+1}. M is *uniform* if every non-zero submodule of M is essential in M. Then it is proved (Goldie [2]) that in any module M with FGD, there exist non-zero uniform submodules $U_1, U_2, \cdots, U_n$ whose sum is direct and essential in M. The number n is independent of the uniform submodules. This number n is called the *Goldie dimension* of M and denoted by $\dim M$. A submodule A of M is termed *small* in M if $A+H=M$ implies $H=M$ for any other submodule H of M. M is called *hollow* if every proper submodule of M is small in M. A module M has *finite spanning dimension* (abbr. FSD) if for every strictly decreasing sequence $H_0, H_1, \cdots$ of submodules of M there is an integer i such that for every $k \geqslant i$, H_k is small in M. A family $\{M\}_{i \in I}$ of submodules of M is said to be a *direct system* if, for any finite number of elements $i_1, i_2, \cdots, i_k$ of I, there is an element i_0 in I such that $M_{i_0} \supseteq M_{i_1} + \cdots + M_{i_k}$. A family $\{M_i\}_{i \in I}$ of submodules of M is said to be an *inverse system* if, for any finite number of elements $i_1, i_2, \cdots, i_k$ of I, there is an element i_0 in I such that $M_{i_0} \subseteq M_{i_1} \cap \cdots \cap M_{i_k}$.

We are now introducing two notions *E-direct system* and *S-inverse system*. A family $\{M_i\}_{i \in I}$ of submodules of M is said to be an E-direct system if for any finite number of elements $i_1, i_2, \cdots, i_k$ of I there is an element i_0 in I such that $M_{i_0} \supseteq M_{i_1} + M_{i_2} + \cdots + M_{i_k}$ and M_{i_0} is non-essential submodule of M. A family $\{M_i\}_{i \in I}$ of submodules of M is said to be an S-inverse system if for any finite number of elements $i_1, i_2, \cdots, i_k$ of I there is an element i_0 in I such that $M_{i_0} \subseteq M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_k}$ and M_{i_0} is non-small.

Note. (i) A family of submodules $\{M_i\}_{i \in I}$ is an E-direct system if and only if the family is a direct system and each M_i of the family is a non-essential submodule of M.

(ii) A family of submodules of M is an S-inverse system if and only if it is an inverse system and each element of the family is a non-small submodule.

Theorems. The purpose of this note is to prove the following two results.

Theorem 1. *For an R-module M the following two conditions are equivalent:*

(i) *M has* FGD; *and*

(ii) *Every E-direct system of non-zero submodules of M is bounded above by a non-essential submodule of M.*

Theorem 2. (i) *If M has* FSD *then every S-inverse system of submodules of M is bounded below by a non-small submodule of M.*

(ii) *If every S-inverse system of M is bounded below by a non-small and non-hollow submodule then M has* FSD.

Proof of Theorem 1. Suppose first that M has FGD and $n=\dim M$. Then there exist uniform submodules $U_1, U_2, \cdots, U_n$ of M whose sum is direct and essential in M. Suppose M has an E-direct system $\{M_i\}_{i\in I}$ of non-zero submodules of M which is not bounded by any non-essential submodule of M. Then $Z=\sum_{i\in I} M_i$ is an essential submodule of M. Let $1\leq j\leq n$. Since Z is essential in M, $Z\cap U_j\neq 0$ and so there exists a non-zero element $a_j\in Z\cap U_j$. Since $a_j\in Z$ there exists a finite subset I_j of I such that $a_j\in\sum_{i\in I_j} M_i$. This is true for all j $(1\leq j\leq n)$. Therefore there exist elements $a_1, a_2, \cdots, a_n$ of M and finite subsets $I_1, I_2, \cdots, I_n$ of I such that the sum $Ra_1+\cdots+Ra_n$ is contained in $\sum_{i\in J} M_i$ where $J=I_1\cup I_2\cup\cdots\cup I_n$. Since $a_i\in U_i$ for $1\leq i\leq n$, the sum $Ra_1+Ra_2+\cdots+Ra_n$ is essential and so $\sum_{i\in J} M_i$ is an essential submodule of M. Since $\{M_i\}_{i\in I}$ is an E-direct system and J is a finite subset of I, there exists an i_0 in I such that $\sum_{i\in J} M_i$ is contained in M_{i_0} and M_{i_0} is a non-essential submodule of M. This is a contradiction to the fact that $\sum_{i\in J} M_i$ is an essential submodule of M. This establishes (ii). Now assume (ii), but suppose M is not a module with FGD. Then there exist an infinite number of non-zero submodules of M whose sum is direct. Let $\{B_i\}_{i\in H}$ be the set of all distinct non-zero submodules of M. Consider the family

$$\mathcal{S}=\left\{\{B_i\}_{i\in I} \Big/ \begin{matrix} I \text{ is an infinite subset of } H \text{ such that} \\ \text{the sum } \sum_{i\in I} B_i \text{ is direct} \end{matrix}\right\},$$

which is not empty by our assumption. For any two elements $\{B_i\}_{i\in I}$ and $\{B_i\}_{i\in J}$ of $\mathcal{S}$ we define $\{B_i\}_{i\in I}\leqslant\{B_i\}_{i\in J}$ if and only if $I\subseteq J$. To show $\mathcal{S}$ is inductive let $\{\{B_i\}_{i\in I_s}\}_{s\in A}$ be a chain of elements of $\mathcal{S}$. Now the union of this chain, that is $\{B_i\}_{i\in I}$ where $I=\bigcup_{s\in A} I_s$, is a member of $\mathcal{S}$. For this we have to show that $\sum_{i\in I} B_i$ is direct. Let $b_{i_s}\in B_{i_s}$ for $1\leq s\leq n$, and $i_s\in I$ such that $b_{i_1}+b_{i_2}+\cdots+b_{i_n}=0$. Let $1\leq s\leq n$. Since $i_s\in I=\bigcup_{i\in A} I_i$ there exists $j_s\in A$ such that $i_s\in I_{j_s}$. Since $\{I_i\}_{i\in A}$ is a chain of sets, there is a $k\in A$ such that $I_{j_s}\subseteq I_k$ for all $1\leq s\leq n$. Now for all $1\leq s\leq n$, the B_{i_s} belongs to $\{B_i\}_{i\in I_k}$.

Therefore $\sum_{s=1}^{n} B_{i_s}$ is direct. Now $b_{i_1+b_{i_2}}+\cdots+b_{i_n}=0$ and $b_{i_s}\in B_{i_s}$ implies $b_{i_j}=0$ for $1\leq j\leq n$. Therefore $\sum_{i\in I} B_i$ is direct and hence $\mathcal{S}$ is inductive. By the Zorn's Lemma $\mathcal{S}$ contains a maximal element, say $\{B_i\}_{i\in N}$. Consider the family

$$\mathcal{B}=\{\sum_{j\in J} B_j / J \text{ is a finite subset of } N\}.$$

Now for any finite number of elements $\sum_{j\in J_1} B_j, \cdots, \sum_{j\in J_s} B_j$ of the family $\mathcal{B}$, their sum is contained in $\sum_{j\in J^*} B_j$ where $J^*=J_1\cup J_2\cup\cdots\cup J_s$. Since J^* is finite we have that J^* is a proper subset of N and so $\sum_{i\in J^*} B_i$ is non-essential. Hence the family is an E-direct system. By the assumed condition (ii), the family $\mathcal{B}$ should be bounded above by a non-essential submodule S of M. Since S is non-essential, there exists $x\in H$ such that $S\cap B_x=0$. Since $S\cap B_x=0$ we have that $x\notin N$. Now consider $N^*=N\cup\{x\}$. Then $\{B_i\}_{i\in N^*}$ is an element of $\mathcal{S}$ and $\{B_i\}_{i\in N}<\{B_i\}_{i\in N^*}$, which is a contradiction to the maximality of $\{B_i\}_{i\in N}$ in $\mathcal{S}$. This completes the proof of Theorem.

Before proving our Theorem 2, we prove the follwing Lemma.

Lemma. *Let A be a non-small submodule of M. If every submodule of A is small in M then A is hollow.*

Proof. Suppose A is not hollow. Then there exist two proper submodules K and L of A such that $K+L=A$. Since A is non-small there exists a proper submodule X of M such that $A+X=M$. Now $K+L+X=M$. Since K is small in M we have $L+X=M$. Similarly since L is small in M we have $X=M$, a contradiction to the fact that X is a proper submodule of M.

Proof of Theorem 2. (i) Suppose M has FSD. Let $\{M_i\}_{i\in I}$ be an S-inverse system of submodules of M. If $\{M_i\}_{i\in I}$ is not bounded below by a non-small submodule, then $\bigcap_{i\in I} M_i$ is a small submodule. Let $i_1\in I$. Since $\bigcap_{i\in I} M_i$ is small, and M_{i_1} is non-small, there is an $j_1\in I$ such that $M_{i_1}\not\subseteq M_{j_1}$. Now there is an $i_2\in I$ such that $M_{i_2}\subseteq M_{i_1}\cap M_{j_1}$. Again since $\bigcap_{i\in I} M_i$ is small and M_{i_2} is non-small there is an $j_2\in I$ such that $M_{i_2}\not\subseteq M_{j_2}$. Now there is an $i_3\in I$ such that $M_{i_3}\subseteq M_{i_2}\cap M_{j_2}$. If we continue this process, we get a strictly decreasing sequence $M_{i_1}, M_{i_2}, \cdots$ of non-small submodules of M, a contradiction to the fact M has FSD. This completes the proof of the part.

(ii) If M has no FSD there is a strictly decreasing infinite chain $M_1, M_2, \cdots$ of non-small submodules of M. Let $\{M_i\}_{i\in B}$ be the set of all distinct non-small submodules of M. Consider

$$\mathcal{I}=\left\{\{M_i\}_{i\in J} \middle/ \begin{array}{l} J \text{ is an infinite subset of } B \text{ and } \{M_i\}_{i\in J} \text{ is a chain} \\ \text{with respect to the set theoretic inclusion} \end{array}\right\},$$

which is not empty by our assumption. For any two elements $\{M_i\}_{i\in J}$ and $\{M_i\}_{i\in J^*}$ of $\mathcal{I}$ we define $\{M_i\}_{i\in J}\leqslant\{M_i\}_{i\in J^*}$ if and only if $J\subseteq J^*$. To show $\mathcal{I}$ is inductive, let $\{\{M_i\}_{i\in J_s}\}_{s\in A}$ be a chain of elements from $\mathcal{I}$. Then the union of this chain, that is $\{M_i\}_{i\in I}$ where $I=\bigcup_{s\in A} J_s$, is also in $\mathcal{I}$. For this consider M_{i_1}, M_{i_2} where $i_1, i_2\in I$. Then $i_1\in J_{s_1}, i_2\in J_{s_2}$ for some $s_1, s_2\in A$. Since $\{J_i\}_{i\in A}$ is also a chain of sets under the set theoretic inclusion, without loss of generality we may suppose that $J_{s_1} \subseteq J_{s_2}$. This implies M_{i_1}, M_{i_2} are members of $\{M_i\}_{i \in J_{s_2}}$. Since $\{M_i\}_{i \in J_{s_2}}$ is an element of $\mathcal{J}$ we have either $M_{i_1} \subseteq M_{i_2}$ or $M_{i_2} \subseteq M_{i_1}$. Therefore $\{M_i\}_{i \in I}$ is an element in $\mathcal{J}$. Hence $\mathcal{J}$ is inductive. Now by the Zorn's Lemma there exist a maximal element, say $\{M_i\}_{i \in J^*}$. Since $\{M_i\}_{i \in J^*}$ is a chain of non-small submodules, it is also an S-inverse system and so $\cap_{i \in J^*} M_i$ is a non-small and non-hollow submodule of M. Since $\cap_{i \in J^*} M_i$ is non-small and non-hollow by the above Lemma $\cap_{i \in J^*} M_i$ properly contains a submodule H' which is non-small in M. Now $H' = M_x$ for some fixed $x \in B$. Since $\cap_{i \in J^*} M_i$ properly contains M_x we have that $x \notin J^*$. Now consider $I^* = J^* \cup \{x\}$. Clearly $\{M_i\}_{i \in I^*} > \{M_i\}_{i \in J^*}$ and $\{M_i\}_{i \in I^*}$ is an element of $\mathcal{J}$, a contradiction to the maximality of $\{M_i\}_{i \in J^*}$. Therefore M has FSD. This completes the proof of Theorem.

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