80. *Some Oscillation Criteria for Second Order Nonlinear Ordinary Differential Equations with Damping*

By Yutaka NAGABUCHI and Minoru YAMAMOTO
Department of Applied Physics, Faculty of Engineering, Osaka University

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1. Introduction. In this paper we consider the oscillatory behavior of the solutions of the second order nonlinear differential equation with damping

(1) $$(r(t)x')' + p(t)x' + q(t)f(x) = 0, \qquad t \in [0, \infty),$$

where $r, p, q \in C[0, \infty)$, $r(t)>0$, and p, q are allowed to take on negative values for arbitrarily large t, $f \in C(\boldsymbol{R})$, $xf(x)>0$ for $x \neq 0$. We restrict our attention to solutions of (1) which exist on some interval $[\tau_0, \infty)$.

For the second order linear differential equation:

(*) $$x'' + q(t)x = 0,$$

the well-known theorem of Wintner [3] for the equation (*) to be oscillatory. Later more general theorems were established by considering weighted averages of the integral of q.

Recently, by the use of completing square and averaging technique, Yan [2] gave the following oscillation theorem for the equation:

(2) $$(r(t)x')' + p(t)x' + q(t)x = 0, \qquad t \in [0, \infty).$$

Theorem. *If there exist* $\alpha \in (1, \infty)$ *and* $\beta \in [0, 1)$ *such that*

(3) $$\overline{\lim_{t\to\infty}}\, t^{-\alpha} \int_{t_0}^{t} (t-\tau)^{\alpha} \tau^{\beta} q(\tau) d\tau = \infty,$$

(4) $$\overline{\lim_{t\to\infty}} \int_{t_0}^{t} [(t-\tau)p(\tau)\tau + \alpha\tau - \beta(t-\tau)]^2 (t-\tau)^{\alpha-2} \tau^{\beta-2} dt < \infty,$$

then (1) *is oscillatory.*

Moreover Yan [1] established two theorems as criteria for the oscillation of (2) when (3) or (4) is not satisfied.

We extend his results for (2) in [1] to the equation (1).

2. Main results. We consider the equation (1) under the following assumption.

Assumption. (a) $r, p,$ and q are continuous on $[0, \infty)$, and $r(t)>0$.

(b) $f: \boldsymbol{R} \to \boldsymbol{R}$ is continuously differentiable such that $xf(x)>0$ $(x \neq 0)$, and $f'(x) \geq k > 0$ for some constant k. Our results are as follows:

Theorem 1. *Suppose that there exist a positive continuously differentiable function* $h(t)$ *on* $[0, \infty)$ *and a constant* $\alpha \in (1, \infty)$ *such that*

(5) $$\overline{\lim_{t\to\infty}}\, t^{-\alpha} \int_{t_0}^{t} H_k(t, \tau) d\tau = \infty,$$

where $H_k(t, \tau) = (t-\tau)^{\alpha} h(\tau) q(\tau)$

$$-\frac{1}{4k}\left[(t-\tau)\frac{h(\tau)p(\tau)}{r(\tau)}+\alpha h(\tau)-(t-\tau)h'(\tau)\right]^2(t-\tau)^{\alpha-2}\frac{r(\tau)}{h(\tau)},$$

then the equation (1) *is oscillatory.*

Theorem 2. *Suppose that there exist a positive continuously differentiable function* $h(t)$ *on* $[0, \infty)$ *and an* $\alpha \in (1, \infty)$ *such that*

$$\overline{\lim_{t\to\infty}}\, t^{-\alpha}\int_s^t H_k(t,\tau)d\tau<\infty, \tag{6}$$

and there exists a continuous function $\varphi(t)$ *on* $[0, \infty)$ *such that*

$$\underline{\lim_{t\to\infty}}\, t^{-\alpha}\int_s^t H_k(t,\tau)d\tau\geq\varphi(s), \tag{7}$$

and

$$\lim_{t\to\infty}\int_0^t \frac{\varphi^+(\tau)^2}{h(\tau)r(\tau)}d\tau=\infty, \tag{8}$$

where $\varphi^+(t)=\max(\varphi(t), 0)$, *then the equation* (1) *is oscillatory.*

Remark. Let $f(x)\equiv 1$ and $k=1$ in (1), the above Theorem 1 and Theorem 2 imply Yan's Theorems in [1].

3. Proofs. *Proof of Theorem* 1. Assume the contrary, then there exists a solution $x(t)$ which may be assumed to be positive on $[t_0, \infty)$ for some $t_0\geq 0$.

Let $\omega(t)=r(t)x'(t)/f(x(t))$, for $t\geq t_0$, then it follows from (1) that

$$\omega'(t)+(f'(x(t))/r(t))\omega(t)^2+(p(t)/r(t))\omega(t)+q(t)=0, \qquad t\geq t_0, \tag{9}$$

Hence, for all $t\geq s\geq t_0$,

$$\int_s^t (t-\tau)^\alpha h(\tau)\omega'(\tau)d\tau+\int_s^t (t-\tau)^\alpha\frac{h(\tau)f'(x(\tau))}{r(\tau)}\omega(\tau)^2d\tau$$
$$+\int_s^t (t-\tau)^\alpha\frac{h(\tau)p(\tau)}{r(\tau)}\omega(\tau)\,d\tau+\int_s^t (t-\tau)^\alpha h(\tau)q(\tau)\,d\tau=0.$$

Noting that

$$\int_s^t (t-\tau)^\alpha h(\tau)\omega'(\tau)d\tau=\alpha\int_s^t (t-\tau)^{\alpha-1}h(\tau)\omega(\tau)d\tau$$
$$-\int_s^t (t-\tau)^\alpha h'(\tau)\omega(\tau)d\tau-\omega(s)(t-\tau)^\alpha h(s),$$

we obtain

$$\int_s^t (t-\tau)^\alpha h(\tau)q(\tau)d\tau=(t-s)^\alpha h(s)\omega(s)-\int_s^t\frac{(t-\tau)^\alpha h(\tau)f'(x(\tau))}{r(\tau)}\omega(\tau)^2d\tau \tag{10}$$
$$-\int_s^t\left[(t-\tau)\frac{h(\tau)p(\tau)}{r(\tau)}+\alpha h(\tau)-(t-\tau)h'(\tau)\right](t-\tau)^{\alpha-1}\omega(\tau)d\tau.$$

From the assumption (b) it follows that

$$\int_s^t (t-\tau)^\alpha h(\tau)q(\tau)d\tau\leq(t-\tau)^\alpha h(s)\omega(s)-k\int_s^t\frac{(t-\tau)^\alpha h(\tau)}{r(\tau)}\omega(\tau)^2d\tau$$
$$-\int_s^t\left[(t-\tau)\frac{h(\tau)p(\tau)}{r(\tau)}+\alpha h(\tau)-(t-\tau)h'(\tau)\right](t-\tau)^{\alpha-1}\omega(\tau)d\tau$$

and hence

$$\int_s^t\left\{(t-\tau)^\alpha h(\tau)q(\tau)-\frac{1}{4k}\left[(t-\tau)\frac{h(\tau)p(\tau)}{r(\tau)}+\alpha h(\tau)-(t-\tau)h'(\tau)\right]^2(t-\tau)^{\alpha-2}\frac{h(\tau)}{r(\tau)}\right\}d\tau$$

$$\leq (t-s)^{\alpha}h(s)\omega(s)-\int_s^t \left\{\sqrt{k}\,(t-\tau)^{\alpha/2}\left(\frac{h(\tau)}{r(\tau)}\right)^{1/2}\omega(t)\right.$$
$$\left.+\frac{1}{2\sqrt{k}}\left[(t-\tau)\frac{h(\tau)p(\tau)}{r(\tau)}+\alpha h(\tau)-(t-\tau)h'(\tau)\right](t-\tau)^{(\alpha-2)/2}\left(\frac{r(\tau)}{h(\tau)}\right)^{1/2}\right\}^2 d\tau$$
$$\leq (t-s)^{\alpha}h(s)\omega(s).$$

Therefore,

$$\int_s^t H_k(t,\tau)d\tau \leq (t-s)^{\alpha}h(s)\omega(s), \qquad s\geq t_0. \tag{11}$$

Divide (11) by t^{α} and take the upper limit as $t\to\infty$, which contradicts the assumption (b). This completes the proof.

Proof of Theorem 2. Let $x(t)$ be a solution of (1). Without loss of generality, we may assume $x(t)\neq 0$ on $[t_0,\infty)$ for some $t_0\geq 0$. Define

$$\omega(t)=r(t)x'(t)/f(x(t)), \qquad t\geq t_0.$$

As in the proof of Theorem 1, it follows that

$$\int_s^t H_k(t,\tau)d\tau \leq (t-s)^{\alpha}h(s)\omega(s), \qquad s\geq t_0. \tag{11}$$

Divide (11) by t^{α} and take the lower limit as $t\to\infty$, we have

$$\varphi(s)\leq h(s)\omega(s), \qquad s\geq t_0,$$

and hence we obtain

$$\varphi^+(s)^2\leq h(s)^2\omega(s)^2, \qquad s\geq t_0. \tag{12}$$

Now we define $u(t)$ and $v(t)$ as follows:

$$u(t)=t^{-\alpha}\int_s^t\left[(t-\tau)\frac{h(\tau)p(\tau)}{r(\tau)}+\alpha h(\tau)-(t-\tau)h'(\tau)\right](t-\tau)^{\alpha-1}\omega(\tau)d\tau$$

$$v(t)=t^{-\alpha}\int_s^t (t-\tau)^{\alpha}h(\tau)\frac{f'(x(\tau))}{r(\tau)}\omega(\tau)^2 d\tau.$$

From (10)

$$u(t)+v(t)=h(s)\omega(s)\left(1-\frac{s}{t}\right)^{\alpha}-t^{-\alpha}\int_s^t (t-\tau)^{\alpha}h(\tau)q(\tau)d\tau.$$

According to (7),

$$\varliminf_{t\to\infty} t^{-\alpha}\int_s^t (t-\tau)^{\alpha}h(\tau)q(\tau)d\tau\geq\varphi(s), \qquad s\geq t_0, \tag{13}$$

and

$$\begin{aligned}&\varlimsup_{t\to\infty} t^{-\alpha}\int_s^t (t-\tau)^{\alpha}h(\tau)q(\tau)d\tau\\ &-\varliminf_{t\to\infty}\frac{t^{-\alpha}}{4k}\int_s^t\left[(t-\tau)\frac{h(\tau)p(\tau)}{r(\tau)}+\alpha h(\tau)-(t-\tau)h'(\tau)\right]^2\\ &\times(t-\tau)^{\alpha-2}\frac{r(\tau)}{h(\tau)}d\tau\geq\varphi(s), \qquad s\geq t_0.\end{aligned} \tag{14}$$

From (6) and (13),

$$\varliminf_{t\to\infty}\frac{t^{-\alpha}}{4k}\int_s^t\left[(t-\tau)\frac{h(\tau)p(\tau)}{r(\tau)}+\alpha h(\tau)-(t-\tau)h'(\tau)\right]^2(t-\tau)^{\alpha-2}\frac{r(\tau)}{h(\tau)}d\tau<\infty.$$

This implies that there exists a sequence $\{t_n\}$ such that

$$t_n\geq t_0, \quad \varlimsup_{n\to\infty} t_n=\infty \quad \text{and}$$

$$\lim_{n\to\infty}\frac{t_n^{-\alpha}}{4k}\int_s^{t_n}\left[(t_n-\tau)\frac{h(\tau)p(\tau)}{r(\tau)}+\alpha h(\tau)-(t_n-\tau)h'(\tau)\right]^2(t_n-\tau)^{\alpha-2}\frac{r(\tau)}{h(\tau)}d\tau<\infty. \tag{15}$$

Taking the upper limit as $t\to\infty$ in (14) and using (13), we obtain

$$\varlimsup_{t\to\infty}\{u(t)+v(t)\}=h(s)\omega(s)-\varliminf_{t\to\infty}t^{-\alpha}\int_s^t(t-\tau)^\alpha h(\tau)q(\tau)d\tau$$
$$\leq h(s)\omega(s)-\varphi(s)=a.$$

Hence there exists a sufficiently large N such that for any $n\geq N$,

(16) $$u(t_n)+v(t_n)<a.$$

Considering the assumption (b), we have

(17) $$v(t)=\int_s^t\left(1-\frac{\tau}{t}\right)^\alpha h(\tau)\frac{f'(x(\tau))}{r(\tau)}\omega(\tau)^2d\tau\geq k\int_s^t\left(1-\frac{\tau}{t}\right)^\alpha h(\tau)\frac{\omega(\tau)}{r(\tau)}d\tau>0,$$

and we observe easily that $v(t)$ is strictly increasing in $t\geq s$. Now suppose that $\lim_{t\to\infty}v(t)=\infty$ and by (16),

(18) $$\lim_{n\to\infty}u(t_n)=-\infty.$$

(16) and (18) imply that for an arbitrarily positive constant η $(0<\eta<1)$, there exists a sufficiently large number N' such that for any $n\geq N'$,

(19) $$u(t_n)/v(t_n)<\eta-1<0.$$

On the other hand, by the Schwartz inequality

$$0\leq t_n^{-2\alpha}\left\{\int_s^{t_n}\left[(t_n-\tau)\frac{h(\tau)p(\tau)}{r(\tau)}+\alpha h(\tau)-(t_n-\tau)h'(\tau)\right]^2(t_n-\tau)^{\alpha-1}\frac{r(\tau)}{h(\tau)}d\tau\right\}$$
$$\leq\left\{t_n^{-\alpha}\int_s^{t_n}\left[(t_n-\tau)\frac{h(\tau)p(\tau)}{r(\tau)}+\alpha h(\tau)-(t_n-\tau)h'(\tau)\right]^2(t_n-\tau)^{\alpha-2}\frac{r(\tau)}{h(\tau)}d\tau\right\}$$
$$\times\left\{t_n^{-\alpha}\int_s^{t_n}(t_n-\tau)^\alpha\frac{h(\tau)}{r(\tau)}\omega(\tau)^2d\tau\right\}.$$

Hence noting (17), for any $n\geq N'$,

$$0\leq u(t_n)^2/v(t_n)$$
$$\leq\frac{t_n^{-\alpha}}{k}\int_s^{t_n}\left[(t_n-\tau)\frac{h(\tau)p(\tau)}{r(\tau)}+\alpha h(\tau)-(t_n-\tau)h'(\tau)\right]^2(t_n-\tau)^{\alpha-2}\frac{r(\tau)}{h(\tau)}d\tau,$$

and by (15) we have

$$\lim_{n\to\infty}(u(t_n)^2/v(t_n))<\infty,$$

which contradicts (18) and (19). Therefore we obtain $\lim_{t\to\infty}v(t)=c<\infty$. From (12) it follows that

$$\lim_{t\to\infty}t^{-\alpha}\int_s^t(t-\tau)^\alpha\frac{\varphi^+(\tau)^2}{h(\tau)r(\tau)}d\tau$$
$$\leq\frac{1}{k}\lim_{t\to\infty}t^{-\alpha}\int_s^t(t-\tau)^\alpha\frac{h(\tau)f'(x(\tau))}{r(\tau)}\omega(\tau)^2d\tau=\frac{1}{k}\lim_{t\to\infty}v(t)=\frac{c}{k}<\infty,$$

which contradicts (8). This completes the proof of Theorem 2.

References

[1] Jurang Yan: Proc. Amer. Math. Soc., **98**, 276–282 (1986).
[2] ——: ibid., **90**, 277–280 (1984).
[3] A. Wintner: Quart. Appl. Math., pp. 115–117 (1949).