79. On the Isomonodromic Deformation of Certain Pfaffian Systems Associated to Appell's Systems (F_2), (F_3)

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§1. **Introduction.** The purpose of this paper is to derive systems of isomonodromic deformation equations associated to Appell's systems (F_2), (F_3), which we shall present in forms that are transformed to Pfaffian systems.

In 1880, P. Appell, generalizing Gauss' hypergeometric equation to the case of two variables, introduced four systems [1]:

$$(F_1)\begin{cases}\theta(\theta+\theta'+\gamma-1)z-x(\theta+\theta'+\alpha)(\theta+\beta)z=0\\ \theta'(\theta+\theta'+\gamma-1)z-y(\theta+\theta'+\alpha)(\theta'+\beta')z=0,\end{cases}$$

$$(F_2)\begin{cases}\theta(\theta+\gamma-1)z-x(\theta+\theta'+\alpha)(\theta+\beta)z=0\\ \theta'(\theta'+\gamma'-1)z-y(\theta+\theta'+\alpha)(\theta'+\beta')z=0,\end{cases}$$

$$(F_3)\begin{cases}\theta(\theta+\theta'+\gamma-1)z-x(\theta+\alpha)(\theta+\beta)z=0\\ \theta'(\theta+\theta'+\gamma-1)z-y(\theta'+\alpha')(\theta'+\beta')z=0,\end{cases}$$

$$(F_4)\begin{cases}\theta(\theta+\gamma-1)z-x(\theta+\theta'+\alpha)(\theta+\theta'+\beta)z=0\\ \theta'(\theta'+\gamma'-1)z-y(\theta+\theta'+\alpha)(\theta+\theta'+\beta)z=0,\end{cases}$$

where $z=z(x,y)$ is unknown function and $\theta=x\partial/\partial x$, $\theta'=y\partial/\partial y$. It is known that Appell's system (F_1) is transformed into a Pfaffian system on $P_2(C)$:

(1.1) $df=(\sum_{i=1}^{6}A_i(dF_i(x,y)/F_i(y,x)))f$, where $f={}^t(z, x\partial z/\partial x, y\partial z/\partial y)$, $A_i\in gl(3,C)$ and F_i's are the defining equations of singular locus of (F_1). (For example see [2].) B. Klares studied the isomonodromic deformation equation of the completely integrable Pfaffian system associated to this system (1.1) in [3]. It is also known that, as well as (F_1), Appell's systems (F_2), (F_3) are transformed into Pfaffian systems of type:

(1.2) $df=(\sum_{i=1}^{6}A_i(dF_i(x,y)/F_i(x,y)))f$, where $f={}^t(z, x\partial z/\partial x, y\partial z/\partial y, xy\partial^2 z/\partial x\partial y)$, $A_i\in gl(4,C)$ and F_i's are the defining equations of singular locus of the systems $(i=1,2,\cdots,6)$.

The author will present the systems of isomonodromic deformation equations associated to Pfaffian systems (1.2) by the same method as in [3]. Main results of this note are obtained in [4].

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§2. **Pfaffian systems satisfying Appell's systems (F_2), (F_3).** Let p: $(C^3)^*\to P_2(C)$ be a canonical projection and (X,Y,Z) be a homogeneous coordinates on $P_2(C)$ with $x=X/Z$, $y=Y/Z$.

Proposition 1. *Appell's system (F_2) is transformed into the following completely integrable Pfaffian system on $P_2(C)$:*

(2.1) $$df=\omega_2 f, \quad f={}^t(z, x\partial z/\partial x, y\partial z/\partial y, xy\partial^2 z/\partial x\partial y), \quad where$$

$$p^*\omega_2=A_1\frac{dX}{X}+A_2\frac{dY}{Y}+A_3\frac{d(X-Z)}{X-Z}+A_4\frac{d(Y-Z)}{Y-Z}+A_5\frac{d(X+Y-Z)}{X+Y-Z}+A_6\frac{dZ}{Z},$$

$$A_1=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 1-\gamma & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1-\gamma\end{bmatrix},\quad A_2=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1-\gamma' & 0\\ 0 & 0 & 0 & 1-\gamma'\end{bmatrix},$$

$$A_3=\begin{bmatrix}0 & 0 & 0 & 0\\ -\alpha\beta & -\alpha-\beta+\gamma-1 & -\beta & -1\\ 0 & 0 & 0 & 0\\ \alpha\beta\beta' & \beta'(\alpha+\beta-\gamma+1) & \beta\beta' & \beta'\end{bmatrix},\quad A_4=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\alpha\beta' & -\beta' & -\alpha-\beta'+\gamma'-1 & -1\\ \alpha\beta\beta' & \beta\beta' & \beta(\alpha+\beta'-\gamma'+1) & \beta\end{bmatrix},$$

$$A_5=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\alpha\beta\beta' & -\beta'(\alpha+\beta-\gamma+1) & -\beta(\alpha+\beta'-\gamma'+1) & -\alpha-\beta-\beta'+\gamma+\gamma'-2\end{bmatrix},$$

$$A_6=\begin{bmatrix}0 & -1 & -1 & 0\\ \alpha\beta & \alpha+\beta & \beta & 0\\ \alpha\beta' & \beta' & \alpha+\beta' & 0\\ -\alpha\beta\beta' & -\beta\beta' & -\beta\beta' & \alpha\end{bmatrix}.$$

Proposition 2. *Appell's system* (F_3) *is transformed into the following Pfaffian system which is completely integrable on* $\boldsymbol{P}_2(\boldsymbol{C})$:

(2.2) $$df=\omega_3 f, \quad f={}^t(z, x\partial z/\partial x, y\partial z/\partial y, xy\partial^2 z/\partial x\partial y), \quad where$$

$$p^*\omega_3=B_1\frac{dX}{X}+B_2\frac{dY}{Y}+B_3\frac{d(X-Z)}{X-Z}+B_4\frac{d(Y-Z)}{Y-Z}+B_5\frac{d(XY-YZ-ZX)}{XY-YZ-ZX}$$
$$+B_6\frac{dZ}{Z},$$

$$B_1=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 1-\gamma & 0 & -1\\ 0 & 0 & 0 & 1\\ 0 & \alpha'\beta' & 0 & \alpha'+\beta'-\gamma'+1\end{bmatrix},\quad B_2=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1-\gamma & -1\\ 0 & 0 & \alpha\beta & \alpha+\beta-\gamma+1\end{bmatrix},$$

$$B_3=\begin{bmatrix}0 & 0 & 0 & 0\\ -\alpha\beta & -\alpha-\beta+\gamma-1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\quad B_4=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\alpha'\beta' & 0 & -\alpha'-\beta'+\gamma-1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix},$$

$$B_5=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & -\alpha'\beta' & -\alpha\beta & -\alpha-\alpha'-\beta-\beta'+\gamma-1\end{bmatrix},$$

$$B_6=\begin{bmatrix}0 & -1 & 0 & 0\\ \alpha\beta & \alpha+\beta & 0 & -1\\ \alpha'\beta' & 0 & \alpha'+\beta' & -1\\ 0 & \alpha'\beta' & \alpha\beta & \alpha+\alpha'+\beta+\beta'\end{bmatrix}.$$

Remark 1. If we transform (F_4) into the Pfaffian system of type (1.2) using the same basis as in the above propositions, A_i $(i=1,2,\cdots,6)$ can not be constant matrices.

§3. Isomonodromic deformation. Let U be a simply connected open domain in C^2 and (M,π,U) be a trivial analytic fibration, with $\pi^{-1}(u)\cong P_2(C)$ $(u=(u_1,u_2)\in U)$ as its fibre. Suppose that M has an analytic hypersurface S, which has the form $S=\cup_{i=1}^{6}S_i$ as its irreducible decomposition, and suppose that S_i's are defined by $F_i(X,Y,Z)=0$, where F_i's are homogeneous polynomials in (X,Y,Z) of degree 1 or 2 with coefficients holomorphic in u $(i=1,2,\cdots,6)$. In the above situation we parametrize the systems (2.1), (2.2) by $u=(u_1,u_2)$ and consider the following Pfaffian systems respectively:

$$(3.1)\qquad df=\Big(\sum_{j=1}^{6}A_j(u)\frac{dF_j(u,X,Y,Z)}{F_j(u,X,Y,Z)}f,\qquad A_j(u)\in gl(4,C(u))\Big),$$

where

$F_1(u,X,Y,Z)=X,\quad F_2(u,X,Y,Z)=Y,\quad F_3(u,X,Y,Z)=X-u_1Z,$
$F_4(u,X,Y,Z)=Y-u_2Z,\quad F_5(u,X,Y,Z)=2(u_2X+u_1Y-u_1u_2Z)/(u_1+u_2),$
$F_6(u,X,Y,Z)=Z,$

and

$$u=(u_1,u_2)\in U\subset C^2\backslash\{u_1,u_2)\,|\,u_1u_2=0,\,u_1+u_2=0\},$$

$$(3.2)\qquad df=\Big(\sum_{j=1}^{6}B_j(u)\frac{dF_j(u,X,Y,Z)}{F_j(u,X,Y,Z)}f,\qquad B_j(u)\in gl(4,C(u))\Big),$$

where

$F_1(u,X,Y,Z)=X,\quad F_2(u,X,Y,Z)=Y,\quad F_3(u,X,Y,Z)=X-u_1Z,$
$F_4(u,X,Y,Z)=Y-u_2Z,\quad F_5(u,X,Y,Z)=XY-u_1YZ-u_2ZX,$
$F_6(u,X,Y,Z)=Z,$

and

$$u=(u_1,u_2)\in U\subset C^2\backslash\{(u_1,u_2)\,|\,u_1u_2=0\}.$$

If these systems satisfy the completely integrable condition, the isomonodromic deformation equations associated to these systems on $P_2(C)$ coincide with the isomonodromic deformation of the same systems restricted on a projective line in $P_2(C)$, which is in general position for S. (As for the definition of "general position", see [2].) Choose $y=x-1$ as such a projective line in a neighborhood of 0 on $P_2(C)$. Then the above systems (3.1), (3.2) restricted on this line are written as follows:

$$(3.3)\qquad df=\Big(A_1\frac{dx}{x}+A_2\frac{d(x-1)}{x-1}+A_3\frac{d(x-u_1)}{x-u_1}+A_4\frac{d(x-(1+u_2))}{x-(1+u_2)}$$
$$+A_5\frac{d(x-u_1(1+u_2)/(u_1+u_2))}{x-u_1(1+u_2)/(u_1+u_2)}\Big)f,$$

$$(3.4)\qquad df=\Big(B_1\frac{dx}{x}+B_2\frac{d(x-1)}{x-1}+B_3\frac{d(x-u_1)}{x-u_1}+B_4\frac{d(x-(1+u_2))}{x-(1+u_2)}$$
$$+B_5\frac{d(x^2-(1+u_1+u_2)x+u_1)}{x^2-(1+u_1+u_2)x+u_1}\Big)f,$$

respectively. Considering the isomonodromic deformation equations of these systems (3.3), (3.4), we get the following theorems.

Theorem 1. *If the system* (3.1) *is completely integrable, then the system of the isomonodromic deformation equations for* (3.1) *is given by the following system of non-linear partial differential equations*:

$$\begin{cases} dA_1=[A_1, A_6]du_1/u_1+[A_5, A_1]U_1,\ \ dA_2=[A_2, A_6]du_2/u_2+[A_5, A_2]U_2,\ \ dA_3=[A_3, \\ A_6]du_1/u_1+[A_5, A_3]U_1,\ \ dA_4=[A_4, A_6]du_2/u_2+[A_5, A_4]U_2,\ \ dA_5=[A_1, A_5](du_1/u_1 \\ -du_2/u_2)-[A_2, A_5](du_1/u_1-du_2/u_2),\ \ dA_6=0,\ \ \textit{where}\ U_1=du_1/u_1-d(u_1+u_2)/ \\ u_1+u_2),\ U_2=du_2/u_2-d(u_1+u_2)/(u_1+u_2). \end{cases}$$

Theorem 2. *If the system* (3.2) *is completely integrable, then the system of the isomonodromic deformation equations for* (3.2) *is given by the following system of non-linear partial differential equations*:

$$\begin{cases} dB_1=[B_1, B_6]du_1/u_1,\ \ dB_2=[B_2, B_6]du_2/u_2,\ \ dB_3=[B_1, B_3]du_1/u_1+[B_5, B_3](du_1/u_1 \\ +du_2/u_2),\ \ dB_4=[B_2, B_4]du_2/u_2+[B_5, B_4](du_1/u_1+du_2/u_2),\ \ dB_5=[B_1, B_5]du_1/d_1 \\ +[B_3, B_5](du_1/u_1+du_2/u_2),\ \ dB_6=0. \end{cases}$$

Remark 2. $dA_6=0$ and $dB_6=0$ owe to fundamental matrices of solutions of the above Pfaffian systems which are normalized at $U\times\{(X, Y, Z)|X-Y-Z=0, Z=0\}$.

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