72. On the Schur Indices of Certain Irreducible Characters of Simple Algebraic Groups over Finite Fields

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Let G be a connected, reductive linear algebraic group defined over a finite field F_q with q elements of characteristic p and F the corresponding Frobenius endomorphism of G. Let G^F denote the group of F-fixed points of G. In [2] R. Gow initiated, in order to determine the Schur indices of irreducible characters of some finite groups of type G^F, to study rationality-properties of the characters of G^F induced by the linear characters of a Sylow p-subgroup of G^F (also cf. A. Helversen-Passoto [4] and Gow [3]). In [5] we have obtained some general results for a general G^F $(p\neq 2)$. The purpose of this paper is to state some more detailed results when G is a simple algebraic group.

Let G be reductive. Let B and T be respectively an F-stable Borel subgroup of G with the unipotent radical U and an F-stable maximal torus of B. Let R be the set of roots of G with respect to T, R^+ the set of positive roots determined by B and D the set of corresponding simple roots. For each $\alpha\in R$, let U_α denote the corresponding root subgroup of G. Let U_+ be the subgroup of U generated by the U_α, $\alpha\in R^+-D$. There is a permutation ρ on D determined by $FU_\alpha=U_{\rho\alpha}$ for $\alpha\in D$. Let I be the set of orbits of ρ on D. For each $i\in I$, put $U_i=\prod_{\alpha\in i}U_\alpha$. Then we have $U/U_+=\prod_{i\in I}U_i$; this decomposition is F-stable and we have $(U/U_+)^F=U^F/U_+^F=\prod_{i\in I}U_i^F$. It is known that U^F is a Sylow p-subgroup of G^F and that if p is a good prime for G then U_+^F is equal to the commutator subgroup of U^F. Let Λ be the set of characters of U^F such that $\lambda|U_+^F=1$ and let Λ_0 be the set of λ in Λ such that $\lambda|U_i^F\neq 1$ for all $i\in I$. Then it is known that, for any $\lambda\in\Lambda_0$, $\Gamma_\lambda=\mathrm{Ind}_{U^F}^{G^F}(\lambda)$ is multiplicity-free ([1], Theorem 8.1.3; also see [5], Lemma 1). For an irreducible character χ of a finite group and a field E of characteristic zero, let $m_E(\chi)$ denote the Schur index of χ with respect to E. We have seen in [5] that if χ is an irreducible character of G^F such that $\langle\chi,\lambda^{G^F}\rangle_{G^F}=1$ for some $\lambda\in\Lambda$ or that, when p is a good prime for G, $p\nmid\chi(1)$, then we have $m_Q(\chi)\leqq 2$, where Q is the field of rational numbers.

Assume now that G is simple. Let $X=\mathrm{Hom}\,(T,G_m)$ be the (additive) module of rational characters of T. Let $P(R)$ and $Q(R)=\langle R\rangle_Z$ be respectively the weight-lattice and the root-lattice of R, where Z is the ring of rational integers. Then we have $P(R)\supset X\supset Q(R)$; and $P(R)/Q(R)$ is a finite group. Put $d=(X:Q(R))$. For an integer n, let $\mathrm{ord}_2\,n$ denote the exponent of the 2-part of n. Then our first result is the following:

Theorem 1. *Let G be a simple algebraic group defined over F_q and assume that $p\neq 2$. Let χ be any irreducible character of G^F such that $\langle\chi, \lambda^{G^F}\rangle_{G^F}=1$ for some $\lambda\in\Lambda$ or that, when p is a good prime for G, $p\nmid\chi(1)$. Then, in any one of the following cases, we have $m_Q(\chi)=1$:* (i) *G is adjoint*; (ii) *G is of type A_l where* $2|l(l+1)/d$ *or* $\mathrm{ord}_2 d>\mathrm{ord}_2(p-1)$; (iii) *$G$ is of type ${}^2A_l$ where* $2|l(l+1)/d$; (iv) *G is of type B_l where* $4|l(l+1)$; $G=\mathrm{Spin}_{2l}^+$ *where either* (a) $4|l(l-1)$ *or* (b) $\mathrm{ord}_2(l-1)=1$ *and* $p\equiv -1 \pmod 4$; (v) $G=SO_{2l}^+$; (vi) $G=H\,\mathrm{Spin}_{2l}$ *where* $4|l$; (vii) $G=\mathrm{Spin}_{2l}^-$ *where* $4|l(l-1)$; (viii) $G=SO_{2l}^-$; (ix) $G={}^3D_4$; (x) *G is of type E_6*; (xi) *G is of type ${}^2E_6$. Moreover, in any one of the following cases, we have $m_{Q_r}(\chi)=1$ for any rational prime $r\neq p$ and we have $m_Q(\chi)=1$ if χ is trivial on Z^F, where Z is the centre of G:* (α) *q is an even power of p*; (β) *G is of type ${}^2A_l$ where q is an odd power of p and* $\mathrm{ord}_2 d>\mathrm{ord}_2(p+1)$; ($\gamma$) *$G$ is of type ${}^2D_l$ where either* (a) $\mathrm{ord}_2 l=1$ *or* (b) $\mathrm{ord}_2(l-1)=1$ *and* $p\equiv 1 \pmod 4$.

Remark. M. J. J. Barry has shown that, for $G={}^3D_4$, p odd, we have $m_Q(\chi)=1$ for any irreducible character χ of G^F.

As to the group $SU_n(F_q)$, it is known that any irreducible character of this group has the Schur index ≤ 2 over $\boldsymbol{Q}$ (Gow [3], Theorem 2.9; the assumption there that p and q are sufficiently large can be removed in virtue of the validity of Ennola-duality for all p, q, which is a result of Hotta-Springer, Kazhdan, Lusztig and Kawanaka). We have

Theorem 2. *Let χ be any irreducible character of $SU_n(F_q)$ where we assume that q is an even power of $p\neq 2$. Then, for any prime number $r\neq p$, we have $m_{Q_r}(\chi)=1$.*

Finally, we state some sufficient conditions to the effect that G^F has an irreducible character of index 2. Let Z be as before the centre of G (simple), and let $\eta_1, \cdots, \eta_c$ be all the distinct irreducible characters of Z^F ($c=|Z^F|$). For $\lambda\in\Lambda_0$ and for $1\leq i\leq c$, put $\Gamma_{\lambda,i}=\mathrm{Ind}_{Z^F U^F}^{G^F}(\eta_i\lambda)$. Then it is easy to show that each $\Gamma_{\lambda,j}$ is multiplicity-free and $\Gamma_\lambda=\sum_{i=1}^{c}\Gamma_{\lambda,j}$. We have (cf. Gow [2, 3]):

Theorem 3. *Let G be a simply-connected, simple algebraic group which is defined and split over F_q, q odd. Then, in any one of the following cases, each $\Gamma_{\lambda,i}$ contains an irreducible character of index 2 over $\boldsymbol{Q}$:* (i) *G is of type A_l where either* (a) *q is an even power of p and* $1\leq\mathrm{ord}_2(l+1)\leq\mathrm{ord}_2(p-1)$, *or,* (b) *$q$ is an odd power of p,* $\mathrm{ord}_2(l+1)=1$ *and* $p\equiv 1 \pmod 4$; (ii) *G is of type B_l where* $4\nmid l(l+1)$ *and either* (a) *q is an even power of p or* (b) *q is an odd power of* $p\equiv 1 \pmod 4$; (iii) *G is of type C_l where either* (a) *q is an even power of p or* (b) *q is an odd power of* $p\equiv 1 \pmod 4$; (iv) *G is of type D_l where either* (a) $\mathrm{ord}_2 l=1$ *and q is an even power of p, or,* (b) $\mathrm{ord}_2 l=1$ *and q is an odd power of* $p\equiv 1 \pmod 8$, *or,* (c) $\mathrm{ord}_2(l-1)=1$ *and q is an even power of* $p\equiv 1 \pmod 4$; (v) *G is of type E_7 where either* (a) *q is an even power of p or* (b) *q is an odd power of* $p\equiv 1 \pmod 4$. *If q is an even power of p, the primes of $\boldsymbol{Q}$ at which the local indices of an irreducible* *constituent* χ *of* $\Gamma_{\lambda,i}$ *can differ from* 1 *are* ∞ *and* p; *if* q *is an odd power of* $p\equiv 1 \pmod 4$, *the only primes of* $\boldsymbol{Q}(\sqrt{p})$ *at which the local indices of* χ *can differ from* 1 *are the real ones.*

Now let us give a brief outline of the proofs. B^F acts on Λ (resp. on Λ_0) by $\lambda^b(u)=\lambda(bub^{-1})$ for $b\in B^F$, $\lambda\in\Lambda$ (resp. $\lambda\in\Lambda_0$) and $u\in U^F$. Let Π be the Galois group of $\boldsymbol{Q}(\zeta_p)$ over $\boldsymbol{Q}$ where ζ_p is a primitive p^{th} root of unity. We shall assume that $p\neq 2$. Fixing one $\lambda\in\Lambda_0$, set $M=\{b\in B^F \mid \lambda^b=\lambda^{\tau(b)}$ for some $\tau(b)\in\Pi\}$; the group M is independent of the choice of $\lambda\in\Lambda_0$. We investigate the rationality of λ^M, $\lambda\in\Lambda$. We have $M=LU^F$ with $L=M\cap T^F$ $(\supset Z^F)$ and $M/Z^FU^F=L/Z^F$; the mapping $b\to\tau(b)$ induces an isomorphism of L/Z^F onto a subgroup of Π (i.e. $\tau(M)$), so that, if α is a fixed generator of $\tau(M)$ and f is an element of L such that $\tau(f)=\alpha$, we have $M=\langle f\rangle Z^FU^F$ and $\langle f \bmod Z^F\rangle \simeq \langle\alpha\rangle$ via τ. For $1\leqq i\leqq c$, put $\mu_i=\mathrm{Ind}_{Z^FU^F}^{M}(\eta_i\lambda)$. Then the μ_i are mutually different irreducible characters of M and we have $\lambda^M=\mu_1+\cdots+\mu_c$. Let $k=\boldsymbol{Q}(\lambda^M)$, the field generated over $\boldsymbol{Q}$, by the values of λ^M, and, for $1\leqq i\leqq c$, let $k_i=k(\mu_i)=k(\eta_i)$. For $1\leqq i\leqq c$, let A_i be the simple direct summand of the group algebra $k_i[M]$ of M over k_i. We see that A_i is isomorphic over k_i to the cyclic algebra $(k_i(\zeta_p), \alpha_i, \eta_i(f^t))$ over k_i, where α_i is a generator of the Galois group of $k_i(\zeta_p)$ over k_i such that $\alpha_i \mid \boldsymbol{Q}(\zeta_p)=\alpha$ and $t=(L:Z^F)$. In order to calculate the Hasse invariants of each A_i, we must therefore determine the structure of M explicitly.

In order to do so we argue as follows. Clearly, it suffices to calculate an element f and the group Z^F. Let $X=\mathrm{Hom}\,(T, G_m)$ be as before. F acts on X by $(F\chi)(t)=\chi(F(t))$ for $\chi\in X$, $t\in T$. We have $F(\rho\alpha)=q\alpha$ for $\alpha\in D$, and D is a basis of $Q(R)=\langle R\rangle_Z$. The way of the action ρ on D is well-known. Therefore, if a basis $\chi_1,\cdots,\chi_l$ of X is suitably chosen (l=rank of G), the way of the action of F on X will be stated explicitly in terms of the χ_i. Thus we get the structure of M completely. It remains to carry out the actual calculation. We note that such a calculation is done implicitly in Carter [1], pp. 39–41.

Theorem 3 follows from rationality-properties of the $\Gamma_{\lambda,i}$, $\lambda\in\Lambda_0$, by an argument similar to the proof of Theorem 3.8 of Gow [3] using the fact that, for $1\leqq i, j\leqq c$, we have $\langle\Gamma_{\lambda,i},\Gamma_{\lambda,j}\rangle_{G^F}=\delta_{ij}\{r(q-1)+1\}/c$ for some integer $r>0$.

References

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