71. *Eigenvalues and Eigenvectors of Supermatrices*

By Yuji KOBAYASHI*) and Shigeaki NAGAMACHI**)

(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1988)

§1. **Introduction and preliminaries.** The theories of linear algebra and analysis over a Grassmann algebra have been developed and are a base of the theory of supermanifolds, Lie supergroups and Lie superalgebras, which are extensively used in modern physics. In his excellent book [1], Berezin treated diagonalization of supermatrices, but he proved it only in a direct way using induction on the number of generators of Grassmann algebras. In this note we study the eigenvalue problem of supermatrices in a general and natural manner by introducing the notions of (super) eigenvalue and eigenvector. We need to consider odd eigenvectors as well as even ones, and corresponding to them two kinds of eigenvalues appear. Starting with the ordinary eigenvalues of the body of a given supermatrix we can find its supereigenvalues by the perturbation method. Our method gives an efficient algorithm to compute eigenvalues and eigenvectors, and we demonstrate this by a simple example. The diagonalization of supermatrices will be done as a by-product of the solution of the eigenvalue problem.

Let Λ be a Grassmann algebra over the complex numbers $\boldsymbol{C}$, generated by a finite or infinite number of odd elements. The algebra Λ is a direct sum of the even part Λ_0 and the odd part Λ_1. The body of an element a of Λ is denoted by $\tilde{a}$. Then the $\tilde{}$ is a mapping of Λ to $\boldsymbol{C}$.

Let p and q be nonnegative integers and let $n=p+q$. By an even (resp. odd) vector we mean a column $(x_1, \cdots, x_p, x_{p+1}, \cdots, x_{p+q})^T$, where x_i is in Λ_0 (resp. Λ_1) for $i=1, \cdots, p$ and in Λ_1 (resp. Λ_0) for $i=p+1, \cdots, p+q$. We consider a supermatrix M of the form $M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A (resp. D) is a $p\times p$-matrix (resp. $q\times q$-matrix) whose entries are in Λ_0 and B (resp. C) is a $p\times q$-matrix (resp. $q\times p$-matrix) whose entries are in Λ_1. If x is an even (resp. odd) vector, then Mx is an even (resp. odd) vector.

A supernumber $\lambda\in\Lambda_0$ is called an *eigenvalue* of a supermatrix M, if there exists a vector x such that $Mx=\lambda x$ and $\tilde{x}=(\tilde{x}_1, \cdots, \tilde{x}_{p+q})^T$ is nonzero. This vector x is called an *eigenvector* corresponding to λ. If x is even (resp. odd), we say λ is an eigenvalue of the first (resp. second) kind.

§2. **Eigenvalues of unmixed matrices.** In this section we consider the case where $p=0$ or $q=0$, and therefore the supermatrices are usual matrices over Λ_0. Let $f(X)=a_0+a_1X+a_2X^2+\cdots+a_nX^n\in\Lambda_0[X]$ be a poly-

*) Department of Mathematics, Tokushima University.
**) Technical College, Tokushima University.

nomial over Λ_0. The body $\tilde{f}(X)$ of $f(X)$ is defined to be the polynomial $\tilde{a}_0+\tilde{a}_1X+\tilde{a}_2X^2+\cdots+\tilde{a}_nX^n \in \boldsymbol{C}[X]$ over $\boldsymbol{C}$.

Lemma 2.1. *Let $f(X)\in\Lambda_0[X]$ be a monic polynomial of degree n. Suppose that the body $\tilde{f}(X)$ is separable and $\alpha_1,\cdots,\alpha_n$ are its roots in $\boldsymbol{C}$. Then $f(X)$ has exactly n roots $\beta_1,\cdots,\beta_n$ in Λ_0, and $\tilde{\beta}_i=\alpha_{\pi(i)}$ for some permutation π of degree n.*

Proof. We will construct the exact root β of $f(X)$ from a root α of $\tilde{f}(X)$ by the Newton method. Let $\alpha^{(0)}=\alpha$ and define

$$\alpha^{(k)}=\alpha^{(k-1)}-f(\alpha^{(k-1)})f'(\alpha^{(k-1)})^{-1} \qquad \text{for } k\geqq 1,$$

where $f'(X)$ is the derivative of $f(X)$. Since α is a simple root of $\tilde{f}(X)$, $\tilde{f}'(\alpha)\neq 0$ and $f'(\alpha^{(0)})$ is invertible. Inductively we see that $f'(\alpha^{(k-1)})$ is invertible, and $\alpha^{(k)}$ above is well defined. Put $\delta_k=f(\alpha^{(k)})$. Then we have

$$\begin{aligned}\delta_k&=f(\alpha^{(k-1)}-f(\alpha^{(k-1)})f'(\alpha^{(k-1)})^{-1})\\&=f(\alpha^{(k-1)})-f'(\alpha^{(k-1)})f(\alpha^{(k-1)})f'(\alpha^{(k-1)})^{-1}\\&\quad+\frac{1}{2}f''(\alpha^{(k-1)})[f(\alpha^{(k-1)})f'(\alpha^{(k-1)})^{-1}]^2+\cdots\\&=\delta_{k-1}^2g(\alpha^{(k-1)})\end{aligned}$$

for some $g(X)\in\Lambda_0(X)$. Since $\tilde{\delta}_0=\tilde{f}(\alpha)=0$, δ_0 is nilpotent. Therefore $\delta_k=0$ for sufficiently large k, and $\beta=\alpha^{(k)}$ is a root of $f(X)$. Moreover $\tilde{\beta}=\tilde{\alpha}^k=\tilde{\alpha}^{(k-1)}=\cdots=\alpha$. Thus we find roots $\beta_1,\cdots,\beta_n$ of $f(X)$ such that $\tilde{\beta}_1=\alpha_1,\cdots,\tilde{\beta}_n=\alpha_n$, and we have $f(X)=(X-\beta_1)\cdots(X-\beta_n)$. If β is another root of $f(X)$, then $f(\beta)=(\beta-\beta_1)\cdots(\beta-\beta_n)=0$. We may assume $\tilde{\beta}=\alpha_1$. Then $(\beta-\beta_2)\cdots(\beta-\beta_n)$ has a nonzero body and invertible, and hence $\beta=\beta_1$.

Let $M=(m_{ij})$ be a matrix over Λ_0. We call the matrix $\tilde{M}=(\tilde{m}_{ij})$ over $\boldsymbol{C}$ the *body* of M. Then, the characteristic polynomial $f(X)=\det(XE-M)$ of M is in $\Lambda_0[X]$ and the characteristic polynomial of $\tilde{M}$ is equal to $\tilde{f}(X)$.

Proposition 2.2. *Let $f(X)$ be a characteristic polynomial of M and suppose $\tilde{f}(X)$ is separable. Then $\lambda\in\Lambda_0$ is an eigenvalue of M if λ is a root of $f(X)=0$. Moreover, if x is an eigenvector of M corresponding to λ, then $\tilde{x}$ is an eigenvector of $\tilde{M}$ corresponding to $\tilde{\lambda}$.*

Proof. Let λ be a root of $f(X)$. Then $\tilde{\lambda}$ is a root of $\tilde{f}(X)$ and is an eigenvalue of $\tilde{M}$. Set $N=\lambda E-M=(n_{ij})$, then $\tilde{N}=\tilde{\lambda}E-\tilde{M}$. Since $\tilde{\lambda}$ is a simple root, some cofactor $\tilde{N}_{ij}$ of $\tilde{N}$ is nonzero. Consider the following Laplace expansion of N:

$$\sum_k n_{lk}N_{ik}=\delta_{li}\det N=0, \qquad l=1,\cdots,n.$$

If we put $x=(N_{i1},\cdots,N_{in})^T$, then we have

$$(\lambda E-M)x=Nx=0 \quad\text{and}\quad \tilde{x}\neq 0.$$

Thus λ is an eigenvalue of M and x is its corresponding eigenvector.

Proposition 2.3. *If $\tilde{M}$ has n different eigenvalues $\alpha_1,\cdots,\alpha_n$, then M also has n different eigenvalues $\beta_1,\cdots,\beta_n$ such that $\tilde{\beta}_i=\alpha_i$ for $i=1,\cdots,n$ and there exists an invertible matrix U over Λ_0 such that $U^{-1}MU=\mathrm{diag}(\beta_1,\cdots,\beta_n)$.*

Proof. From Lemma 2.1 and Proposition 2.2, M has eigenvalues

$\beta_1, \cdots, \beta_n$ such that $\tilde{\beta}_i = \alpha_i$ and corresponding eigenvectors $x_1, \cdots, x_n$. Put $U=(x_1, \cdots, x_n)$, then U is invertible because $\tilde{U}=(\tilde{x}_1, \cdots, \tilde{x}_n)$ is invertible. Clearly, $U^{-1}MU = \mathrm{diag}(\beta_1, \cdots, \beta_n)$.

§3. Eigenvalues of supermatrices. In this section we treat general supermatrices given in Section 1.

Theorem 3.1. *Let* $M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be a supermatrix such that the eigenvalues* $\alpha_1, \cdots, \alpha_p$ *of* $\tilde{A}$ *and the eigenvalues* $\delta_1, \cdots, \delta_q$ *of* $\tilde{D}$ *are all different. Then* M *has eigenvalues* $\beta_1, \cdots, \beta_p$ *and* $\gamma_1, \cdots, \gamma_q$ *such that* $\tilde{\beta}_1=\alpha_1, \cdots, \tilde{\beta}_p=\alpha_p$ *and* $\tilde{\gamma}_1=\delta_1, \cdots, \tilde{\gamma}_q=\delta_q$. *Moreover, the eigenvalues* $\beta_1, \cdots, \beta_p$ *(resp.* $\gamma_1, \cdots, \gamma_q$*) are of the first (resp. second) kind, and there exists an invertible supermatrix* U *such that* $U^{-1}MU=\mathrm{diag}(\beta_1, \cdots, \beta_p, \gamma_1, \cdots, \gamma_q)$.

Proof. From Propositon 2.3, there are invertible matrices U_1 and U_2 such that $U_1^{-1}AU_1=\mathrm{diag}(a_1, \cdots, a_p)$ and $U_2^{-1}DU_2=\mathrm{diag}(d_1, \cdots, d_q)$, where $\tilde{a}_i = \alpha_i$ and $\tilde{d}_i=\delta_i$. Let $V=\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$ and $M'=\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}=V^{-1}MV$. Then we can find an eigenvalue $a_1+\mu$ of M' with $\tilde{\mu}=0$ and its corresponding even eigenvector $z_1=(1, x_2, \cdots, x_p, y_1, \cdots, y_q)^T$ as follows. From the equation $M'z_1=(a_1+\mu)z_1$, we have

$$
\begin{aligned}
&B_1'y=\mu, \\
&a_2x_2+B_2'y=a_1x_2+\mu x_2, \\
&\cdots\cdots \\
&a_px_p+B_p'y=a_1x_p+\mu x_p, \qquad (1)\\
&c_c'x+d_1y_1=a_1y_1+\mu y_1, \\
&\cdots\cdots \\
&c_q'x+d_qy_q=a_1y_q+\mu y_q,
\end{aligned}
$$

where B_i' is the i-th row of B', C_i' is the i-th row of C', $x=(1, x_2, \cdots, x_p)^T$ and $y=(y_1, \cdots, y_q)^T$. Since the body of $a_i-a_1-\mu=a_i-a_1-B_1'y$ is nonzero, the first p equations in (1) give

$$x_i=(a_i-a_1-B_1'y)^{-1}B_i'y,$$

for $i=2, \cdots, p$. Thus x_i is a polynomial $f_i(y)$ in anti-commuting variables $y_1, \cdots, y_n$ over Λ, Substituting x_i by $f_i(y)$ and μ by $B_1'y$ in the $(p+1)$-th equation and taking account of the fact that $y_1^2=0$, we get

$$(d_1-a_1+g(y_2, \cdots, y_q))y_1=h(y_2, \cdots, y_q),$$

where g and h are polynomials in $y_2, \cdots, y_q$ over Λ. Since $g(y_2, \cdots, y_q)$ is bodyless and d_1-a_1 has nonzero body, $d_1-a_1+g(y_2 \cdot \cdots, y_q)$ is invertible, and we have

$$y_1=(d_1-a_1+g(y_2, \cdots, y_q))^{-1}h(y_2, \cdots, y_q).$$

Similarly y_j is expressed as a polynomial in $y_{j+1}, \cdots, y_q$ for $2 \leqq j \leqq q$. Hence y_q is written by the entries of M', and the system (1) of equations is solved. Thus we obtain an eigenvalue $\beta_1=a_1+\mu$ of M' and its corresponding eigenvector z_1. Similarly we get eigenvalues $\beta_2, \cdots, \beta_p$ of M' and their corresponding eigenvectors $z_2, \cdots, z_p$.

To obtain an eigenvalue γ_j of the second kind, we solve the equation

$M'w_j=(d_j+\mu)w_j$, where $w_j=(x_1, \cdots, x_p, y_1, \cdots y_{j-1}, 1, y_{j+1}, \cdots, y_q)^T$ is an odd vector. Let $w=(z_1, \cdots, z_p, w_1, \cdots, w_q)$, then w is invertible since $\tilde{W}=E$. Now let $U=VW$, then $U^{-1}MU=\mathrm{diag}(\beta_1, \cdots, \beta_p, \gamma_1, \cdots, \gamma_q)$.

Example 3.2. Our proofs are constructive and give us an algorithm to compute eigenvalues and eigenvectors of a given supermatrix. Now we perform a computation for the case $p=1$ and $q=1$. Let $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a supermatrix such that $a, d \in \Lambda_0$ and $b, c \in \Lambda_1$. Suppose that $\tilde{a} \neq \tilde{d}$. Let us calculate along the method in the proof of Theorem 3.1. From the equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = (a+\lambda) \begin{bmatrix} 1 \\ y \end{bmatrix}$$

we have $by=\lambda$, $c+dy=ay+\lambda y$, where $\lambda \in \Lambda_0$ and $y \in \Lambda_1$. Since $y^2=0$, we have $c+dy=ay$, and the invertibility of $a-d$ gives

$$y=(a-d)^{-1}c, \qquad \lambda=b(a-d)^{-1}c.$$

Thus we get an eigenvalue $a+bc(a-d)^{-1}$ and its corresponding eigenvector $(1, c(a-d)^{-1})^T$. Similarly, we have another eigenvalue $d+bc(a-d)^{-1}$ and its corresponding eigenvector $(b(d-a)^{-1}, 1)^T$. Let

$$U=(a-d)^{-1} \begin{bmatrix} a-d & -b \\ c & a-d \end{bmatrix}.$$

Then

$$U^{-1}=(a-d)^{-2} \begin{bmatrix} (a-d)^2-bc & b(a-d) \\ -c(a-d) & (a-d)^2+bc \end{bmatrix},$$

and we have

$$U^{-1}MU=\begin{bmatrix} a+bc(a-d)^{-1} & 0 \\ 0 & d+bc(a-d)^{-1} \end{bmatrix}.$$

Reference

[1] F. A. Berezin: Introduction to Superanalysis. Reidel Publishing Co., Dordorecht, Boston, Lancaster, Tokyo (1987).