66. The Uniqueness of Periodic Solutions of Liénard Equations in some Domains Including the Origin

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1. Introduction. The existence and the uniqueness of the periodic solutions of Liénard equation:

$$\ddot{x}+f(x)\dot{x}+g(x)=0$$

or equivalently

$$\begin{cases}\dot{x}=y-F(x) \\ \dot{y}=-g(x)\end{cases} \quad \text{where } F(x)=\int_0^x f(u)du,$$

has been widely discussed and numerous criteria have been developed.

Lins, de Melo and Pugh [1] showed that there exists at most one periodic orbit when F is a polynomial of degree 3 and $g(x)\equiv x$. Rychkov [3] proved that if F is an odd polynomial of degree 5 and $g(x)\equiv x$ then there exist at most 2 periodic orbits.

Lloyd [2] investigated the number of periodic solutions when F is polynomial-like and showed that under certain conditions there are at least n periodic solutions if F behaves like a polynomial of degree $2n+1$ or $2n+2$. But the maximal number of periodic solutions has remained unsolved.

In this paper, we give certain conditions for the uniqueness of the periodic solution in some bounded domains including the origin.

2. Lloyd's results. Lloyd assumed in his paper [1]:

[H]
(1) F and g are continuously differentiable,
(2) $xg(x)>0 \quad (x\neq 0)$,
(3) there exist k_i $(i=1\sim 4)$ satisfying

$$k_1<k_2<0<k_3<k_4,$$
$$F(k_1),\ F(k_3)>0 \quad \text{and} \quad F(k_2),\ F(k_4)<0,$$
$$f(x)<0 \quad \text{on } [k_1,k_2]\cup[k_3,k_4],\ f(x)>0 \text{ on } [R_2,R_3].$$

By (2), there is no critical point except for the origin.

For the convenience we write

$a_i=F(k_i)$,

$b_i=G(k_i)$ where $G(x)=\int_0^x g(u)du$ and,

ξ_1, ξ_2 are the zeros of $F(x)$ in $[k_1,k_2]$, $[k_3,k_4]$ respectively.

Under the following conditions:

[C1]
$$\frac{1}{2}a_1^2+b_1-G(\xi_1)\geq\frac{1}{2}(a_3+\sqrt{2b_4})^2,$$
$$\frac{1}{2}a_4^2+b_4-G(\xi_2)\geq\frac{1}{2}(a_3-\sqrt{2b_4})^2,$$

Lloyd proved that the closed curve γ_1 through the points (k_1, c_1) and (k_4, c_4) can be constructed, and that the interior domain D_1 surrounded by γ_1 is negatively invariant. And this result with the asymptotic stability of the origin implies that there exists at least one periodic orbit in D_1. But in his paper [2], Lloyd did not prove the uniqueness of the periodic orbit (not critical) in the domain D_1.

3. Main theorem. We will report that under certain conditions, there exists a unique periodic orbit in D_1 and give the sketch of the proof.

Lemma. *We assume that* [H] *holds. Moreover under the following conditions*:

[C2]
$$\frac{1}{2}a_1^2+b_1\geq 2G(\xi_1), \qquad \frac{1}{2}a_4^2+b_4\geq 2G(\xi_2) \quad \text{and}$$
$$\frac{1}{2}a_2^2+b_2\geq G(\xi_2), \qquad \frac{1}{2}a_3^2+b_3\geq G(\xi_1),$$

the closed curve γ_2 *through the points* $(\xi_1, 0)$, (k_3, a_3), $(\xi_2, 0)$, (k_2, a_2) *can be constructed in* D_1 *and the interior domain* D_2 *surrounded by* γ_2 *is positively invariant.*

Sketch of proof. Let γ_2^+ be the subarc of γ_2 contained in the halfplane $x\geq 0$. γ_2^+ has two sections.

We consider the curve defined by

$$\frac{1}{2}y^2+G(x)=G(\xi_1).$$

This curve goes through $(\xi_1, 0)$. By $(1/2)a_1^2+b_1\geq 2G(\xi_1)$, this curve lies in γ_1, and by $(1/2)a_2^2+b_2\geq G(\xi_2)$, this curve must intersects the curve $y=F(x)$ at $x=\xi^+\in(0,\xi_2)$. We construct one section of γ_2^+ by curve above where $x\in[\xi_1, \xi^+]$.

The other section of γ_2^+ is constructed by $y=F(x)$ where $x\in(\xi^+, \xi_2]$. Similary, γ_2^-, the subarc of γ_2 contained in the halfplane $x\leq 0$, is constructed by

$$\frac{1}{2}y^2+G(x)=G(\xi_2) \quad \text{and} \quad y=F(x).$$

Considering the directions of vector field and evaluating the differentiation along the solutions, the positive invariance of D_2 can be easily proved. Q.E.D.

Theorem. *We assume that* [H] *holds. Under the conditions* [C1] *and* [C2], *there exists a unique periodic orbit* (*except for the critical point*) *in* D_1.

Sketch of proof. [C1] guarantees the existence of periodic orbits in D_1. We will prove the uniqueness.

By the invariance principle (cf., for example [4]), we can show that the solution starting from a point in D_2 is attracted to the origin and that there is no periodic orbit in D_2. This result implies that all the periodic orbits in D_1 must have points in common with both the lines

$$x=\xi_1 \quad \text{and} \quad x=\xi_2.$$

Let C be the innermost periodic orbit. We consider the orbit $\tilde{C}$ starting from a point in D_1 which is on the line $x=\xi_1$, $y>0$, outside of C. If this orbit does not cross the line $x=\xi_1$ in D_1 again then this orbit must not be periodic. So we assume that $\tilde{C}$ crosses the line $x=\xi_1$ again. Moreover, if $\tilde{C}$ goes outside of γ_1 then the negative invariance of D_1 implies that $\tilde{C}$ will remain outside of γ_1 in the future and that $\tilde{C}$ cannot be a periodic orbit. Therefore we can assume that $\tilde{C}$ will cross the line $x=\xi_1$ in D_1 outside of C.

Now, we define $I, \tilde{I}$ by

$$I=\int_C du, \quad \tilde{I}=\int_{\tilde{C}} du \qquad \text{where } u=\frac{1}{2}y^2+G(x).$$

By the fact that $I=0$, if we can show that $I<\tilde{I}$ then it will be proved that $\tilde{C}$ is not periodic and that there is no periodic orbit outside of C in D_1. Dividing $C, \tilde{C}$ into four subarcs as follows:

$C_1, \tilde{C}_1$: the subarcs of $C, \tilde{C}$ corresponding to $x\leq\xi_1$,

$C_2, \tilde{C}_2$: the subarcs of $C, \tilde{C}$ corresponding to $\xi_1\leq x\leq\xi_2$ and $y\geq 0$,

$C_3, \tilde{C}_3$: the subarcs of $C, \tilde{C}$ corresponding to $x\geq\xi_2$,

$C_4, \tilde{C}_4$: the subarcs of $C, \tilde{C}$ corresponding to $\xi_1\leq x\leq\xi_2$ and $y\leq 0$,

respectively. We compare I_i and $\tilde{I}_i$ $(i=1\sim 4)$ where

$$I_i=\int_{C_i} du, \quad \tilde{I}_i=\int_{\tilde{C}_i} du \qquad (i=1\sim 4).$$

For comparing I_2 and $\tilde{I}_2$, we use the expression for du:

$$du=\frac{-g(x)F(x)}{y-F(x)}dx.$$

The curves C_2 and $\tilde{C}_2$ can be regarded as the graphs of $y=y(x)$ and $y=\tilde{y}(x)$ respectively.

$$I_2=\int_{\xi_1}^{\xi_2}\frac{-g(x)F(x)}{y(x)-F(x)}\,dx<\int_{\xi_1}^{\xi_2}\frac{-g(x)F(x)}{\tilde{y}(x)-F(x)}\,dx\leq\tilde{I}_2.$$

Similarly we can prove that $I_4<\tilde{I}_4$.

For comparing I_1 and $\tilde{I}_1$, we use the expression for du:

$$du=F(x)dy.$$

The curves C_1 and $\tilde{C}_1$ can be regarded as the graphs of $x=x(y)$ and $x=\tilde{x}(y)$ respectively.

$$I_1=\int_{y_1}^{y_2}F(x(y))dy\leq\int_{\tilde{y}_1}^{\tilde{y}_2}F(\tilde{x}(y))dy=\tilde{I}_1.$$

Similarly we can prove that $I_3\leq\tilde{I}_3$.

These inequalities show that $I<\tilde{I}$ and the proof is completed. Q.E.D.

Remark 1. Lloyd also considered the case that

$$F(k_1),\ F(k_3)<0 \quad\text{and}\quad F(k_2),\ F(k_4)>0$$

and obtained a similar existence result given in Section 2. On the uniqueness, our methods can be applied in the above case and a similar theorem as above can be proved.

Remark 2. If it is assumed that there exists a bounded solution outside of γ_1 then we have at least one periodic orbit outside of γ_1. But the number of periodic solutions outside of γ_1 has not been decided.

References

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