# 63. *Solvability in Distributions for a Class of Singular Differential Operators.* I[†]

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The local solvability for Fuchsian operators has been studied by many authors (see Tahara [4] and its references). In this paper, the author will establish the local solvability in $\mathcal{D}'$ for a class of (non-Fuchsian) singular hyperbolic operators including

$$L=(t\partial_t)^2-\Delta_x+a(t,x)(t\partial_t)+\langle b(t,x),\partial_x\rangle+c(t,x).$$

§1. **Theorem.** Let us consider

$$P=(t\partial_t)^m+\sum_{\substack{j+|\alpha|\leqq m\\ j<m}} a_{j,\alpha}(t,x)(t\partial_t)^j\partial_x^\alpha,$$

where $(t,x)=(t,x_1,\cdots,x_n)\in \boldsymbol{R}_t\times\boldsymbol{R}_x^n$, $\partial_t=\partial/\partial t$, $\partial_x=(\partial/\partial x_1,\cdots,\partial/\partial x_n)$, $m\in\{1,2,3,\cdots\}$, $\alpha=(\alpha_1,\cdots,\alpha_n)\in\{0,1,2,\cdots\}^n$, $|\alpha|=\alpha_1+\cdots+\alpha_n$ and $\partial_x^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$. On the coefficients, we assume that $a_{j,\alpha}(t,x)$ ($j+|\alpha|\leqq m$ and $j<m$) are $C^\infty$ functions defined in an open neighborhood $U$ of $(0,0)$ in $\boldsymbol{R}_t\times\boldsymbol{R}_x^n$. For $(t,x)\in U$ and $\xi\in\boldsymbol{R}_\xi^n\setminus\{0\}$, denote by $\lambda_1(t,x,\xi),\cdots,\lambda_m(t,x,\xi)$ the roots of the equation (in $\lambda$)

$$\lambda^m+\sum_{\substack{j+|\alpha|=m\\ j<m}} a_{j,\alpha}(t,x)\lambda^j\xi^\alpha=0.$$

Assume that for any $(t,x,\xi)\in U\times(\boldsymbol{R}_\xi^n\setminus\{0\})$ the following conditions (H-1)–(H-3) hold:

(H-1) $\lambda_i(t,x,\xi)$ is real for $1\leqq i\leqq m$.

(H-2) $\lambda_i(t,x,\xi)\neq\lambda_j(t,x,\xi)$ for $1\leqq i\neq j\leqq m$.

(H-3) $\lambda_i(t,x,\xi)\neq 0$ for $1\leqq i\leqq m$.

Then we have:

**Theorem.** *There is an open neighborhood $V$ of $(0,0)$ in $\boldsymbol{R}_t\times\boldsymbol{R}_x^n$ which satisfies the following: for any $f(t,x)$ $(=f)\in\mathcal{D}'(\overline{V})$ there exists a $u(t,x)$ $(=u)\in\mathcal{D}'(\overline{V})$ such that $Pu=f$ holds on $V$.*

Here, $\overline{V}$ denotes the closure of $V$ and $\mathcal{D}'(\overline{V})$ denotes the set of all distributions defined in a neighborhood of $\overline{V}$.

**Remark 1.** In the $C^\infty$ function space, the above operator $P$ was already discussed and the following results are known: (1) the local solvability in $C^\infty$ (by Tahara [3], Serra [2]) and (2) the existence of $C^\infty$ null-solutions (by Mandai [1]).

**Remark 2.** By the same argument given below, we can prove the local solvability in $\mathcal{D}'$ (near the origin) also for the following type of (non-Fuchsian) hyperbolic operators

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[†] Dedicated to Professor Tosihusa KIMURA on his 60th birthday.

$$L_1=(t\partial_t)^2-t^2\partial_{x_1}^2-\partial_{x_2}^2+a(t,x)(t\partial_t)+b_1(t,x)\partial_{x_1}+b_2(t,x)\partial_{x_2}+c(t,x),$$
$$L_2=(t\partial_t)((t\partial_t)^2-\partial_x^2)+\sum_{i+j\leqq 2} a_{i,j}(t,x)(t\partial_t)^i\partial_x^j$$

under $b_1(0,x)=0$ near $x=0$ (for $L_1$) and $a_{0,2}(0,0)\notin\{-1,-2,\cdots\}$ (for $L_2$).

**§2. Proof of Theorem.** As in Tahara [4], Theorem is obtained by the following two propositions.

**Proposition 1.** *Let $P$ be as in §1. Then, there are $s_k>0$ $(k=0,1,2,\cdots)$ and an open neighborhood $U_0$ of $(0,0)$ in $\boldsymbol{R}_t\times\boldsymbol{R}_x^n$ which satisfy the following: for any $k\in\{0,1,2,\cdots\}$, $s>s_k$, an open subset $W$ of $U_0$ and $g\in H^{-k}(W)$ there exists a $v\in H^{-k}(W\cap\{t>0\})$ [resp. $v\in H^{-k}(W\cap\{t<0\})$] such that $P(t^{-s}v)=t^{-s}g$ holds on $W\cap\{t>0\}$ [resp. on $W\cap\{t<0\}$]. Here $H^{-k}$ denotes the usual Sobolev space.*

**Proposition 2.** *Let $P$ be as in §1. Then, there is an open neighborhood $V$ of $(0,0)$ in $\boldsymbol{R}_t\times\boldsymbol{R}_x^n$ which satisfies the following: for any $f\in\mathcal{D}'(\overline{V})$ with $\operatorname{supp}(f)\subset\{t=0\}$ there exists a $u\in\mathcal{D}'(\overline{V})$ with $\operatorname{supp}(u)\subset\{t=0\}$ such that $Pu=f$ holds on $V$.*

In fact, if we assume these facts, we can prove theorem as follows. Let $f\in\mathcal{D}'(\overline{V})$. Then we have $f\in H^{-k}(W)$ for some $k\in\{0,1,2,\cdots\}$ and some open set $W$ satisfying $\overline{V}\subset W\subset U_0$. Take $s\in\boldsymbol{Z}$ such that $s>s_k$. Then by putting $g=t^sf$ and by Proposition 1 we can find $v_+\in H^{-k}(W\cap\{t>0\})$ and $v_-\in H^{-k}(W\cap\{t<0\})$ such that $P(t^{-s}v_+)=f$ on $W\cap\{t>0\}$ and $P(t^{-s}v_-)=f$ on $W\cap\{t<0\}$. Take $u_1\in\mathcal{D}'(W)$ such that $u_1=t^{-s}v_+$ on $W\cap\{t>0\}$ and $u_1=t^{-s}v_-$ on $W\cap\{t<0\}$ (note that this is possible, since $v_+\in H^{-k}(W\cap\{t>0\})$ and $v_-\in H^{-k}(W\cap\{t<0\})$ hold). Put $f_1=f-Pu_1$. Then $f_1\in\mathcal{D}'(W)$ and $\operatorname{supp}(f_1)\subset\{t=0\}$. Therefore by Proposition 2 and the condition $\overline{V}\subset W$ we have $u_2\in\mathcal{D}'(\overline{V})$ such that $\operatorname{supp}(u_2)\subset\{t=0\}$ and that $Pu_2=f_1$ holds on $V$. Thus, by putting $u=u_1+u_2$ we obtain $u\in\mathcal{D}'(\overline{V})$ such that $Pu=f$ holds on $V$.

Hence, to have theorem it is sufficient to prove Propositions 1 and 2 above.

**§3. Proof of Proposition 1.** Define $P_{-s}$ by $P_{-s}u=t^sP(t^{-s}u)$. Then by Lemma 1 given below we can see the following: there are $s_k>0$ $(k=0,1,2,\cdots)$, $\delta_k>0$ $(k=0,1,2,\cdots)$ and an open neighborhood $U_0$ of $(0,0)$ in $\boldsymbol{R}_t\times\boldsymbol{R}_x^n$ such that

$$\|(P_{-s})^*\varphi\|_k^2\geqq\delta_k s^2\|\varphi\|_k^2 \tag{3.1}$$

holds for any $s>s_k$ and $\varphi\in C_0^\infty(U_0\cap\{t>0\})$, where $\|*\|_k$ denotes the norm in the Sobolev space $H^k(U_0\cap\{t>0\})$.

By using this fact, let us show Proposition 1. Let $s>s_k$ and $g\in H^{-k}(W)$. Denote by $H_0^k(W\cap\{t>0\})$ the closure of $C_0^\infty(W\cap\{t>0\})$ in $H^k(W\cap\{t>0\})$. Define a linear subspace $Z$ of $H_0^k(W\cap\{t>0\})$ by $Z=\{(P_{-s})^*\varphi;\ \varphi\in C_0^\infty(W\cap\{T>0\})\}$ and a linear functional $T$ on $Z$ by $T((P_{-s})^*\varphi)=\langle\varphi,g\rangle$. Then by (3.1) we can see that $T$ is well-defined and it is continuous on $Z$ with respect to the topology induced from $H_0^k(W\cap\{t>0\})$. Therefore we can find a $v\in H^{-k}(W\cap\{t>0\})$ such that $T(z)=\langle z,v\rangle$ for any $z\in Z$, that is, $\langle(P_{-s})^*\varphi,v\rangle=\langle\varphi,g\rangle$ for any $\varphi\in C_0^\infty(W\cap\{T>0\})$. Hence, we have $P_{-s}v=g$ on $W\cap\{t>0\}$ and therefore $P(t^{-s}v)=t^{-s}g$ on $W\cap\{t>0\}$.

**Lemma 1.** *Let $A(t,x,D_x)(=A(t))$ be an $m\times m$ matrix of pseudo-differential operators of order* 1 *on $\boldsymbol{R}_x^n$ depending smoothly on $t\in[0,T]$, and let $A(t)^*$ be the formal adjoint operator of $A(t)$. Assume that every component of $A(t)+A(t)^*$ is of order* 0. *Put*

$$L_s=t\partial_t+s+A(t,x,D_x).$$

*Then, for any $k\in\{0,1,2,\cdots\}$ there are $s_k>0$ and $c_k>0$ such that*

$$\|L_s\varphi\|_k^2\geqq c_k s^2\|\varphi\|_k^2$$

*holds for any $s>s_k$ and $\varphi\in C_0^\infty((0,T)\times\boldsymbol{R}_x^n)^m$.*

**§4. Proof of Proposition 2.** Put

$$C(\rho\,;x,\partial_x)=\rho^m+\sum_{\substack{j+|\alpha|\leqq m\\ j<m}}a_{j,\alpha}(0,x)\rho^j\partial_x^\alpha.$$

Then by the fact that $C(\rho\,;x,\partial_x)$ is elliptic near $x=0$ (by (H-3)) and by Lemma 2 given below we can see the following: there are $\delta_k>0$ $(k=0,1,2,\cdots)$ and an open neighborhood $\Omega_0$ of $x=0$ in $\boldsymbol{R}_x^n$ such that

$$\|C(-l\,;x,\partial_x)^*\varphi\|_k^2\geqq\delta_k\|\varphi\|_{k+m}^2$$

holds for any $l\in\{0,1,2,\cdots\}$ and $\varphi\in C_0^\infty(\Omega_0)$, where $\|*\|_k$ denotes the norm in the Sobolev space $H^k(\Omega_0)$. Therefore by taking $\Omega\subset\Omega_0$ we have the following: for any $\mu(x)\in\mathcal{D}'(\bar{\Omega})$ and any $l\in\{0,1,2,\cdots\}$ there exists a $\psi(x)\in\mathcal{D}'(\bar{\Omega})$ such that $C(-l\,;x,\partial_x)\psi=\mu$ holds on $\Omega$.

By using this fact, let us show Proposition 2. Take an open neighborhood $V$ so that $V\cap\{t=0\}(=\Omega)\subset\Omega_0$. Let $f\in\mathcal{D}'(\bar{V})$ be such that $\operatorname{supp}(f)\subset\{t=0\}$. Then $f$ is expressed in the form $f=\sum_{i=0}^N\delta^{(i)}(t)\otimes\mu_i(x)$ for some $\mu_i\in\mathcal{D}'(\bar{\Omega})$. Put $u=\sum_{i=0}^N\delta^{(i)}(t)\otimes\psi_i(x)$. Then we can see that $Pu=f$ is equivalent to the following:

$$(4.1)\qquad\begin{cases}C(-N-1\,;x,\partial_x)\psi_N=\mu_N,\\ C(-N\,;x,\partial_x)\psi_{N-1}=\mu_{N-1}+F_{N-1}(\psi_N),\\ \quad\vdots\\ C(-1\,;x,\partial_x)\psi_0=\mu_0+F_0(\psi_1,\cdots,\psi_N),\end{cases}$$

where $F_i(\psi_{i+1},\cdots,\psi_N)$ is a distribution in $x$ determined by $\psi_{i+1},\cdots,\psi_N$. Therefore by solving (4.1) successively we can obtain $\psi_i\in\mathcal{D}'(\bar{\Omega})$ $(i=0,1,\cdots,N)$ so that (4.1) holds on $\Omega$, that is $Pu=f$ holds on $V$.

**Lemma 2.** *Let $a_i(x,D_x)$ $(i=1,\cdots,m)$ be pseudo-differential operators of order* 1 *with real symbol $a_i(x,\xi)$ satisfying $|a_i(x,\xi)|\geqq c(1+|\xi|)$ on $\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n$ for some $c>0$, and let $B(x,D_x)$ be an $m\times m$ matrix of pseudo-differential operators of order* 0. *Put*

$$K_s=s+\sqrt{-1}\begin{bmatrix}a_1(x,D_x) & & 0\\ & \ddots & \\ 0 & & a_m(x,D_x)\end{bmatrix}+B(x,D_x).$$

*Then, there are $s_0>0$ and $d_k>0$ $(k=0,1,2,\cdots)$ such that*

$$\|K_s\varphi\|_k^2\geqq d_k\|\varphi\|_{k+1}^2$$

*holds for any $s>s_0$ and $\varphi\in C_0^\infty(\boldsymbol{R}_x^n)^m$.*

## References


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