# 50. Zeros of $L(s,\chi)$ in Short Segments on the Critical Line

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1. Let $L(s,\chi)$ be the Dirichlet $L$-function with $\chi$ primitive (mod $k$), $k>1$. Let $N_0(T,\chi)$ be the number of zeros of $L(s,\chi)$ on the segment $s=1/2+it$, $0\leqq t\leqq T$. The purpose of the present note is to give a brief proof of

**Theorem.** *Let $T\geqq k^{(1/2)+6\varepsilon}$, $U\geqq (kT)^{(1/3)+2\varepsilon}$ with small $\varepsilon>0$. Then we have*

$$N_0(T+U,\chi)-N_0(T,\chi)\gg_\varepsilon U\log T.$$

This should be compared with Karatsuba [2], and we stress that a minor modification of our argument can yield a slight improvement upon his result. There are two important ingredients in our argument: One is Atkinson's method [1], and the other is Weil's result [6] on character sums. More specifically, we have combined Selberg's ideas [5] with ours [3]–[4].

2. Here we outline our proof of the theorem. The details will be published elsewhere.

Let $L(s,\chi)=\psi(s,\chi)L(1-s,\bar\chi)$ be the functional equation for $L(s,\chi)$, and put $X(t,\chi)=\psi^{-1/2}(1/2+it,\chi)L(1/2+it,\chi)$ which is real for real $t$. Also, as in [5], let $\alpha(\nu)$ be the coefficient in the Dirichlet series expansion for $\zeta(s)^{-1/2}$, and let $\beta(\nu)=\alpha(\nu)(\log\xi/\nu)/\log\xi$ with $\xi$ to be determined later. We put

$$\eta(t,\chi)=\sum_{\nu<\xi}\chi(\nu)\beta(\nu)\nu^{-(1/2)-it}.$$

And we consider the estimation of

$$I=\int_{-U\log T}^{U\log T}\left|\int_0^H X(T+t+u,\chi)|\eta(T+t+u,\chi)|^2du\right|^2 e^{-(t/U)^2}dt,$$

$$J=\int_{-U\log T}^{U\log T}\left|\int_0^H L\left(\frac{1}{2}+i(T+t+u),\chi\right)\eta^2(T+t+u,\chi)du-H\right|^2 e^{-(t/U)^2}dt,$$

where $H\ll 1$, $(kT)^{(1/3)+2\varepsilon}\leqq U\leqq T^{1-\varepsilon}$.

Invoking the result of [4] we have, as a first step,

$$\begin{aligned} I\ll U\xi^2T^{-\varepsilon}+\int_0^H\int_0^H\left(\frac{kT}{2\pi}\right)^{(i/2)(u-v)}\sum_{\nu<\xi}\frac{\chi(\nu_1\nu_2)\bar\chi(\nu_3\nu_4)}{(\nu_1\nu_2\nu_3\nu_4)^{1/2}}\left(\frac{\nu_3}{\nu_1}\right)^{iu}\left(\frac{\nu_4}{\nu_2}\right)^{iv}\beta(\nu_1)\beta(\nu_2)\beta(\nu_3)\beta(\nu_4)\\ \times\int_{-U\log T}^{U\log T}e^{-(t/U)^2}L\left(\frac{1}{2}+i(T+t+u),\chi\right)L\left(\frac{1}{2}-i(T+t+v),\bar\chi\right)\\ \times\left(\frac{\nu_3\nu_4}{\nu_1\nu_2}\right)^{i(T+t)}dt\,du\,dv. \end{aligned}\tag{1}$$

Then we apply a modified version of Atkinson's splitting argument to this product of values of $L$-functions. For this sake let $a$, $b$ be two positive integers such that $(a,b)=1$ and $(ab,k)=1$. And we write, for $\mathrm{Re}(z)>1$, $\mathrm{Re}(w)>1$,

$$L(z,\chi)L(w,\bar{\chi})=\{\sum_{am=bn}+\sum_{am<bn}+\sum_{am>bn}\}\chi(m)\bar{\chi}(n)m^{-z}n^{-w}.$$

The first sum is $\bar{\chi}(a)\chi(b)a^{-w}b^{-z}L(z+w,\chi_0)$ where $\chi_0$ is the principal character (mod $k$). The other two sums are treated as in [3, V], and we get, for $\mathrm{Re}\,(z)<1$, $\mathrm{Re}\,(w)<1$,

$$\begin{aligned}(2)\quad & L(z,\chi)L(w,\bar{\chi})=\bar{\chi}(a)\chi(b)a^{-w}b^{-z}L(z+w,\chi_0)\\ &+\bar{\chi}(a)\chi(b)k^{1-z-w}a^{z-1}b^{w-1}\Gamma(z+w-1)\zeta(z+w-1)\prod_{p|k}(1-p^{z+w-2})\\ &\times\left\{\frac{\Gamma(1-w)}{\Gamma(z)}+\frac{\Gamma(1-z)}{\Gamma(w)}\right\}+\bar{\chi}(a)\chi(b)a^{z-1}b^{w-1}(g_{a,b}(z,w;\chi)+g_{b,a}(w,z;\bar{\chi})).\end{aligned}$$

Here

$$\begin{aligned}(3)\quad & g_{a,b}(z,w;\chi)=\chi(a)a^{1-z}\{\Gamma(z)\Gamma(w)(e^{2\pi iz}-1)(e^{2\pi iw}-1)\}^{-1}\sum_{c=0}^{b-1}\sum_{m,n=1}^{k}\chi(m)\bar{\chi}(am+n)\\ &\times\int_C\int_C\frac{x^{z-1}y^{w-1}e^{-n(y-2\pi ic/b)}}{1-e^{-k(y-2\pi ic/b)}}\left\{\frac{e^{-m(x+ay-2\pi iac/b)}}{1-e^{-k(x+ay-2\pi iac/b)}}-\frac{\delta(c)}{k(x+ay)}\right\}dxdy,\end{aligned}$$

where $\delta(c)=1$ if $c=0$ and $\delta(c)=0$ if $c\neq 0$, and the contour $C$ is as in [3]. In (2) we set $z=1/2+i(T+t+u)$, $w=1/2-i(T+t+v)$, $a=\nu_1\nu_2/(\nu_1\nu_2,\nu_3\nu_4)$, $b=\nu_3\nu_4/(\nu_1\nu_2,\nu_3\nu_4)$, and insert it into (1). The contribution to $I$ of the first two terms on the right of (2) can be estimated as in [5], and we see that it is $\ll UH^{3/2}(\log\xi)^{-1/2}$, providing $(\log\xi)^{-1}\leqq H\leqq(\log\xi)^{-1/2}$. Hence, for such $H$ we have

$$\begin{aligned}(4)\quad & I\ll U\xi^2T^{-\varepsilon}+UH^{3/2}(\log\xi)^{-1/2}+\sum_{\substack{\nu<\xi\\(\nu,k)=1}}(\nu_1\nu_2\nu_3\nu_4)^{-1}\\ &\times\int_0^H\int_0^H\left|\int_{-\infty}^{\infty}e^{-(t/U)^2}g_{a,b}\left(\frac{1}{2}+i(T+t+u),\frac{1}{2}-i(T+t+v),\chi\right)dt\right|dudv,\end{aligned}$$

where $a$, $b$ are as above.

On the other hand, when $\mathrm{Re}\,(z)<0$, $\mathrm{Re}\,(w)>1$, we may deduce, from (3),

$$g_{a,b}(z,w;\chi)=h_{a,b}(z,w;\chi)+\overline{h_{a,b}(\bar{z},\bar{w};\bar{\chi})};$$

$$h_{a,b}(z,w;\chi)=\sum_{n=1}^{\infty}\sigma_{1-z-w}(n,\chi;ab)e^{-2\pi i\widetilde{ak}n/b}\int_0^{\infty}x^{-z}(1+x)^{-w}e^{-2\pi inx/abk}dx,$$

$$\sigma_{1-z-w}(n,\chi;ab)=k^{-1}\sum_{fg=n}g^{1-z-w}\sum_{m=1}^{k}\chi(m)\bar{\chi}(m+g)e^{2\pi i\overline{ab}mf/k},$$

where $ak\widetilde{ak}\equiv 1$ (mod $b$) and $ab\overline{ab}\equiv 1$ (mod $k$). Then we have to find an analytic continuation of $h_{a,b}\,(z,1-z-i\tau;\chi)$ which is defined for $Re(z)<0$ and real $\tau$. This is accomplished, as in [3, II], by computing the truncated Voronoi formula for the sum

$$A(x)=\sum_{n\leq x}\sigma_{i\tau}(n,\chi;ab)e^{-2\pi i\widetilde{ak}n/b},$$

which yields, uniformly for $X\geqq 1$, $\tau\ll 1$, and arbitrary $a$, $b$,

$$\int_X^{2X}|A(x)|^2dx\ll_\varepsilon X^{3/2}+(kX)^{1+\varepsilon},$$

and thus a continuation of $h_{a,b}(z,1-z-i\tau;\chi)$ to $\mathrm{Re}\,(z)<3/4$. With this we may follow closely the argument of [3]–[4], and show that the infinite integral in (4) is $\ll abT^{\varepsilon}(kT/U)^{1/2}$. Namely we have

$$I\ll UH^{3/2}(\log\xi)^{-1/2}+U\xi^2T^{-\varepsilon}+\xi^2T^{\varepsilon}(kT/U)^{1/2}.$$

In much the same way we can show the same estimate for $J$. Then, choosing $\xi = T^{c(\varepsilon)}$ appropriately we obtain the upper bound $\ll UH^{3/2}(\log \xi)^{-1/2}$ for both $I$ and $J$, providing $(kT)^{(1/3)+2\varepsilon} \leqq U \leqq T^{1-\varepsilon}$. The rest of the proof is much the same as the corresponding part of [5].

## References

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