46. Symmetrization of the van der Corput Generalized Sequences

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(Communicated by Shokichi IYANAGA, M. J. A., May 12, 1988)

1. Introduction. Let $\sigma=(x_n)_0^\infty$ be an infinite sequence in the unit interval $E=[0,1]$. The sequence σ is called *uniformly distributed* in E if $\lim_{N\to\infty} A_N(\sigma;x)=x$ for all $x\in E$, where $A_N(\sigma;x)/N$ denotes the number of terms x_n, $0\leqq n\leqq N-1$, which are less than x. The *diaphony* $F_N(\sigma)$ and the L^2 *discrepancy* $T_N(\sigma)$ of the sequence σ are defined for every positive integer N as follows:

$$F_N(\sigma)=(2\textstyle\sum_{h=1}^\infty (1/h^2)\,|(1/N)S_N(\sigma;h)|^2)^{1/2}$$

and

$$T_N(\sigma)=\left(\int_0^1 |A_N(\sigma;x)/N-x|^2\,dx\right)^{1/2},$$

where

$$S_N(\sigma;h)=\textstyle\sum_{n=0}^{N-1}\exp(2\pi i h x_n)$$

is the exponential sum of σ. It is well known (see [9] and [10]), that both $T_N(\sigma)\to 0$ and $F_N(\sigma)\to 0$ are equivalent to the sequence σ being uniformly distributed in E. Also it is known (see [5] and [6]), that the best possible order of magnitude of both $T_N(\sigma)$ and $F_N(\sigma)$ is $N^{-1}(\log N)^{1/2}$.

Now let $(r_j)_1^\infty$ be a given infinite sequence of integers $r_j\geqq 2$. Suppose also that for every integer $j\geqq 0$ we are given a permutation τ_j of the set $\{0,1,\cdots,r_{j+1}-1\}$. For the sake of brevity, we put $R_0=0$ and $R_j=r_1r_2\cdots r_j$ for $j\geqq 1$. The *van der Corput generalized sequence* $\sigma=(\varphi(n))_0^\infty$, associated with the given sequences $(r_j)_1^\infty$ and $(\tau_j)_0^\infty$, was constructed by Faure [2] as follows: For an integer $n\geqq 0$, let

$$n=\textstyle\sum_{j=0}^\infty a_jR_j \qquad (a_j\in\{0,1,\cdots,r_{j+1}-1\},\ j=0,1,\cdots)$$

be the (r_j)-adic expansion of n. Then set

$$\varphi(n)=\textstyle\sum_{j=0}^\infty \tau_j(a_j)/R_{j+1}.$$

In the present paper, we prove that if the sequence $(r_j)_1^\infty$ satisfies the condition $\sum_{j=1}^n r_j^2=O(n)$, then both the diaphony $F_N(\sigma)$ of the van der Corput generalized sequence σ and the L^2 discrepancy $T_N(\tilde\sigma)$ of any symmetric sequence $\tilde\sigma$ produced by σ have the best possible order of magnitude $N^{-1}(\log N)^{1/2}$. Also we obtain an exact estimate for the L^2 discrepancy of a class of two-dimensional finite sequences associated with the van der Corput generalized sequences.

2. Statement of the results.

Theorem 1. *Suppose that $(r_j)_1^\infty$ satisfies the condition*

(1) $$\textstyle\sum_{j=1}^n r_j^2\leqq Bn \qquad \text{for all } n\in N,$$

where B is a positive constant. Then, for every integer $N \geqq 1$, the diaphony of the van der Corput generalized sequence σ satisfies

$$NF_N(\sigma) \leqq \pi C(r, B)\,(\log rN)^{1/2}, \tag{2}$$

where $r=\min\{r_j \mid j \in N\}$ and $C(r, B)=((B-1)/(3\log r))^{1/2}$.

In order to formulate the next two theorems we need the notion of symmetric sequence (see [5]). A sequence $(y_n)_0^\infty$ in E is called *symmetric* if $y_{2n}+y_{2n+1}=1$ for every $n \geqq 0$. A symmetric sequence $(y_n)_0^\infty$ is said to be *produced by an infinite sequence* $(x_n)_0^\infty$ if for every integer $n \geqq 0$ we have either $y_{2n}=x_n$ or $y_{2n+1}=x_n$. Obviously, every infinite sequence in E produces at least one symmetric sequence.

Theorem 2. *Suppose that $(r_j)_1^\infty$ satisfies* (1). *Let $\tilde{\sigma}$ be any symmetric sequence produced by the van der Corput generalized sequence σ. Then for every integer $N \geqq 1$, we have*

$$NT_N(\tilde{\sigma}) \leqq C(r, B)\,(\log(rN/2))^{1/2}+1, \tag{3}$$

where r and $C(r, B)$ are defined as in the previous theorem.

Now let X be a finite sequence consisting of N points in the unit square E^2. Then the L^2 *discrepancy* $T(X)$ of X is defined by

$$T(X)=\left(\iint_{E^2} |A(x, y)/N-xy|^2\,dxdy\right)^{1/2},$$

where $A(x, y)$ denotes the number of points of X lying in the rectangle $[0, x) \times [0, y)$. From the well known theorems of Roth [8] and Devenport [1], it follows that the best possible order of magnitude of $T(X)$ is also $N^{-1}(\log N)^{1/2}$. In the next theorem, we construct a very large family of finite sequences in E^2 whose L^2 discrepancy has the best possible order of magnitude.

Theorem 3. *Suppose again that $(r_j)_1^\infty$ satisfies* (1). *Let $\tilde{\sigma}=(y_n)_0^\infty$ be any symmetric sequence in E produced by the van der Corput generalized sequence σ, and let $N \geqq 1$ be a given integer. Then for the L^2 discrepancy $T(X_N)$ of the two-dimensional finite sequence X_N consisting of the points*

$$(n/N, y_n), \qquad n=0, 1, \cdots, N-1,$$

we have

$$NT(X_N) \leqq C(r, B)\,(\log(rN/2))^{1/2}+2, \tag{4}$$

where r and $C(r, B)$ are the same as in Theorem 1.

Let $\tilde{\sigma}=(y_n)_0^\infty$ be a symmetric sequence in E produced by the van der Corput generalized sequence σ, and let $N \geqq 2$ be an integer. From [7: Theorem A], it follows that $NT_N(\tilde{\sigma}) \leqq (1/\pi)nF_n(\sigma)+1$, where $n=[N/2]$. (Here $[x]$ denotes the integral part of a real x.) From this and Theorem 1 we immediately obtain Theorem 2. Further, from one-dimensional case of [6: Theorem 1], it follows that there exists an integer n with $1 \leqq n \leqq N$ such that $NT(X_N) \leqq nT_n(\tilde{\sigma})+1$. From this and Theorem 2 we get Theorem 3. Hence, we have to prove only Theorem 1. A sketch of its proof is given in Sections 3 and 4.

Remark 1. If $(r_j)_1^\infty$ satisfies a stronger condition than (1), then the estimates (2), (3) and (4) admit a minor improvement. For example, if $r_j^2 \leqq B$ for $j \geqq 1$, then $\log rN$ in (2) can be replaced by $\log((r-1)N+1)$.

Remark 2. We note that in the special case $r_1=r_2=\cdots=r$ and $\tau_0=\tau_1=\cdots=I$, where $r\geqq 2$ is an integer and I is the identical permutation of the set $\{0,1,\cdots,r-1\}$, the above results are due to Proinov and Grozdanov [7].

Remark 3. In connection with Theorem 2 we shall formulate a result of Faure [3]. Consider the symmetric sequence

$$\tilde{\sigma}=(\varphi(0),1-\varphi(0),\varphi(1),1-\varphi(1),\cdots)$$

and put

$$c=\overline{\lim}_{N\to\infty} NT_N(\tilde{\sigma})/(\log N)^{1/2}.$$

In the special case $r_1=r_2=\cdots=2$ and $\tau_0=\tau_1=\cdots=I$, Faure proved that $0.29\cdots\leqq c<0.34$, and conjectured that $c=0.29\cdots$.

3. Auxiliary results. In this section we do not suppose that $(r_j)_1^\infty$ satisfies (1). Let σ be the van der Corput generalized sequence.

Lemma 1. *The sequence σ has the following two properties:*

(i) *For every integer $n\geqq 0$ there exists a real number β_n such that* $\{\varphi(j)\mid j=0,1,\cdots,R_n-1\}=\{j/R_n+\beta_n\mid j=0,1,\cdots,R_n-1\}$.

(ii) *For all integers a,b and n with $0\leqq a<r_{n+1}$, $0\leqq b<R_n$ and $n\geqq 0$, we have*

$$\varphi(aR_n+b)=\varphi(aR_n)+\varphi(b)-\varphi(0).$$

We omit the proof of Lemma 1 since it can easily be verified directly by the definition of $\varphi(n)$. We see moreover that (i) holds for

$$\beta_n=\sum_{j=n}^{\infty}\tau_j(0)/R_{j+1}.$$

Lemma 2. *Let $N=aR_n+b$, where a,b and n are integers with $1\leqq a<r_{n+1}$, $1\leqq b\leqq R_n$ and $n\geqq 0$. Then the exponential sum $S_N(\sigma;h)$ of σ satisfies*

$$|S_N(\sigma;h)|\leqq|S_{aR_n}(\sigma;h)|+|S_b(\sigma;h)| \qquad \textit{for all } h\in \boldsymbol{Z}.$$

Proof. Let $h\in\boldsymbol{Z}$. Using Lemma 1-(ii) we deduce

$$S_N(\sigma;h)=S_{aR_n}(\sigma;h)+S_b(\sigma;h)\exp(2\pi ih(\varphi(aR_n)-\varphi(0))),$$

which implies the desired inequality. Q.E.D.

Lemma 3. *Let $N\geqq 1$ be an integer, and let*

$$N=\sum_{j=0}^{n} a_jR_j \qquad (a_j\in\{0,1,\cdots,r_{j+1}-1\},\ j=0,1,\cdots) \tag{5}$$

be its (r_j)-adic expansion. Then for the exponential sum $S_N(\sigma;h)$ of σ, we have the estimate

$$|S_N(\sigma;h)|\leqq\sum_{j=0}^{n} a_jR_j\delta_{R_j}(h) \qquad \textit{for all } h\in\boldsymbol{Z}, \tag{6}$$

where

$$\delta_m(h)=\begin{cases}1 & \textit{if } h\equiv 0 \pmod{m},\\ 0 & \textit{if } h\not\equiv 0 \pmod{m}.\end{cases}$$

Proof. Let $h\in\boldsymbol{Z}$. First we state that

$$|S_{aR_n}(\sigma;h)|\leqq aR_n\delta_{R_n}(h) \tag{7}$$

for every integer a with $1\leqq a\leqq r_{n+1}$. Indeed, from Lemma 2, it follows that

$$|S_{aR_n}(\sigma;h)|\leqq a\,|S_{R_n}(\sigma;h)| \tag{8}$$

for the same values of a. From Lemma 1-(i) and the well known identity $m\delta_m(h)=\sum_{j=0}^{m-1}\exp(2\pi ihj/m)$, we get

$$S_{R_n}(\sigma;h)=R_n\delta_{R_n}(h)\exp(2\pi ih\beta_n). \tag{9}$$

From (8) and (9), we obtain the desired inequality (7). Further, we may assume with no loss of generality that $a_n \neq 0$ in (5). Now to complete the proof of (6) one can use induction on n and the inequality (7). Q.E.D.

Remark 4. Let $N \geqq 1$ and $h \geqq 1$ be integers. Then Lemma 3 implies that $|S_N(\sigma; h)| \leqq r_n h - 1$, where n satisfies $R_{n-1} \leqq h < R_n$. From this and the well known Weyl criterion for uniform distribution (see [4: p. 7]), we conclude that every van der Corput generalized sequence is uniformly distributed in E.

4. Proof of Theorem 1. Let $N \geqq 1$ be a given integer, and let (5) be its (r_j)-adic expansion with $a_n \neq 0$. From the definition of the diaphony $F_N(\sigma)$ and Lemma 3, we get

$$\begin{aligned} N^2 F_N^2(\sigma) &\leqq 2 \sum_{j=0}^{n} \sum_{\nu=0}^{n} a_j a_\nu R_j R_\nu \sum_{h=1}^{\infty} (1/h^2) \delta_{R_j}(h) \delta_{R_\nu}(h) \\ &= 4 \sum_{j=0}^{n} \sum_{\nu=0}^{j} a_j a_\nu R_j R_\nu \sum_{h=1}^{\infty} (1/h^2) \delta_{R_j}(h) \\ &\quad - 2 \sum_{j=0}^{n} a_j^2 R_j^2 \sum_{h=1}^{\infty} (1/h^2) \delta_{R_j}(h) \\ &= (\pi^2/3) \sum_{j=0}^{n} (2 a_j R_j^{-1} \sum_{\nu=0}^{j} a_\nu R_\nu - a_j^2) \\ &\leqq (\pi^2/3) \sum_{j=0}^{n} a_j (2 r_{j+1} - a_j) \\ &\leqq (\pi^2/3) \sum_{j=0}^{n} (r_{j+1}^2 - 1). \end{aligned}$$

From this and (1) we obtain

$$N^2 F_N^2(\sigma) \leqq (\pi^2/3)(B-1)(n+1). \tag{10}$$

On the other hand, it follows from (5) that $N \geqq R_n \geqq r^n$, and so $n \leqq (\log N)/(\log r)$. Hence, (10) implies the desired estimate (2). Q.E.D.

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