39. On Fundamental Solution of Differential Equation with Time Delay in Banach Space

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This paper is concerned with the fundamental solution in the sense of S. Nakagiri [4] to the linear differential equation with time delay

$$(1)\qquad \frac{d}{dt}u(t)=Au(t)+A_1u(t-r)+\int_{-r}^{0}a(\tau)A_2u(t+\tau)d\tau+f(t)$$

in a Banach space X. We assume that

(i) A is a densely defined closed linear operator which generates an analytic semigroup $T(t)$ in X.

(ii) A_1 and A_2 are closed linear in general unbounded operators with domains $D(A_1)$ and $D(A_2)$ containing the domain $D(A)$ of A.

(iii) a is a uniformly Hölder continuous real valued function in $[-r,0]$, where r is some fixed positive number.

For the sake of convenience we assume that A has an everywhere defined bounded inverse.

The solvability of the initial value problem for the equation (1) as well as fundamental results on the semigroup associated with it was established by G. Di Blasio, K. Kunisch, and E. Sinestrary [1], [2], [3], [11] under a mild smoothness hypothesis on the coefficient a in the delay term, i.e. $a\in L^1(-r,0)$ or $a\in L^2(-r,0)$.

The fundamental solution $W(t)$ to the equation (1) is by definition a bounded operator valued function satisfying

$$W(t)=\begin{cases}T(t)+\int_0^t T(t-s)\{A_1W(s-r)+\int_{-r}^{0}a(\tau)A_2W(s+\tau)d\tau\}ds, & t\geqq 0\\ 0 & t<0.\end{cases}$$

With the aid of the change of the variable $\tau\to\tau-s$ and noting that $W(t)=0$ for $t<0$ we get

$$(2)\qquad W(t)=T(t)+\int_0^t T(t-s)\int_0^s a(\tau-s)A_2W(\tau)d\tau ds$$

in $[0,r]$, and

$$(3)\qquad \begin{aligned}W(t)=T(t)&+\int_r^t T(t-s)A_1W(s-r)ds\\&+\int_0^t T(t-s)\int_{s-r}^{s}a(\tau-s)A_2W(\tau)d\tau ds\end{aligned}$$

in (r,∞). The exchange of the order of integration yields

$$(4)\qquad W(t)=T(t)+\int_0^t\int_\tau^t T(t-s)a(\tau-s)dsA_2W(\tau)d\tau$$

in $[0,r]$, and

$$
(5)\quad \begin{aligned} W(t)=T(t)+\int_r^t T(t-s)A_1W(s-r)ds+\int_0^{t-r}\int_\tau^{\tau+r} T(t-s)a(\tau-s)dsA_2W(\tau)d\tau \\ +\int_{t-r}^t\int_\tau^t T(t-s)a(\tau-s)dsA_2W(\tau)d\tau \end{aligned}
$$

in (r, ∞). It will turn out that $AW(t)$ has singularities at $t=nr$, $n=0, 1, 2, \cdots$, and

$$\|AW(t)\| \leqq C_n/(t-nr), \qquad nr<t\leqq(n+1)r.$$

Hence, the right sides of (2)–(5) should be interpreted in the improper sense: in $(0, r]$

$$
\begin{aligned} W(t)&=T(t)+\lim_{\varepsilon\to+0}\int_\varepsilon^t T(t-s)\int_\varepsilon^s a(\tau-s)A_2W(\tau)d\tau \\ &=T(t)+\lim_{\varepsilon\to+0}\int_\varepsilon^t\int_\tau^t T(t-s)a(\tau-s)dsA_2W(\tau)d\tau, \end{aligned}
$$

and in $n<t\leqq(n+1)r$, $n=1, 2, \cdots$,

$$
\begin{aligned} W(t)=T(t)+\lim_{\varepsilon\to+0}\Bigg[&\left(\sum_{j=1}^{n-1}\int_{jr+\varepsilon}^{(j+1)r}+\int_{nr+\varepsilon}^t\right)T(t-s)A_1W(s-r)ds \\ &+\sum_{j=0}^{n-1}\int_{jr}^{(j+1)r}T(t-s)\left(\int_{s-r}^{jr}+\int_{jr+\varepsilon}^s\right)a(\tau-s)A_2W(\tau)d\tau ds \\ &+\int_{nr}^t T(t-s)\left(\int_{s-r}^{nr}+\int_{nr+\varepsilon}^s\right)a(\tau-s)A_2W(\tau)d\tau ds\Bigg] \\ =T(t)+\lim_{\varepsilon\to+0}\Bigg[&\left(\sum_{j=1}^{n-1}\int_{jr+\varepsilon}^{(j+1)r}+\int_{nr+\varepsilon}^t\right)T(t-s)A_1W(s-r)ds \\ &+\left(\sum_{j=0}^{n-2}\int_{jr+\varepsilon}^{(j+1)r}+\int_{(n-1)r+\varepsilon}^{t-r}\right)\int_\tau^{\tau+r}T(t-s)a(\tau-s)dsA_2W(\tau)d\tau \\ &+\left(\int_{t-r}^{nr}+\int_{nr+\varepsilon}^t\right)\int_\tau^t T(t-s)a(\tau-s)dsA_2W(\tau)d\tau\Bigg]. \end{aligned}
$$

Theorem 1. *The fundamental solution $W(t)$ to* (1) *exists and is unique. $W(t)$ is differentiable at $t\neq nr$, $n=0, 1, 2, \cdots$ and satisfies*

$$\frac{d}{dt}W(t)=AW(t)+A_1W(t-r)+\int_{-r}^0 a(\tau)A_2W(t+\tau)d\tau.$$

The functions $dW(t)/dt$ and $AW(t)$ are strongly continuous in $(nr, (n+1)r]$, $n=0, 1, 2, \cdots$, and the following inequalities hold: for $n=0, 1, 2, \cdots$

$$
\left.\begin{aligned} &\|AW(t)\|\leqq C_n/(t-nr) \\ &\|dW(t)/dt\|\leqq C_n/(t-nr) \\ &\|AW(t)A^{-1}\|\leqq C_n \end{aligned}\right\} \qquad nr<t\leqq(n+1)r,
$$

$$
\left.\begin{aligned} &\|A(W(t')-W(t))\|\leqq C_{n,\alpha}(t'-t)^\alpha(t-nr)^{-\alpha-1} \\ &\|A(W(t')-W(t))A^{-1}\|\leqq C_{n,\alpha}(t'-t)^\alpha(t-nr)^{-\alpha} \end{aligned}\right\} \qquad \begin{aligned} &nr<t<t'\leqq(n+1)r, \\ &0<\alpha<\rho, \end{aligned}
$$

$$\left\|\int_{nr}^t AW(\tau)d\tau\right\|=\left\|\lim_{\varepsilon\to+0}\int_{nr+\varepsilon}^t AW(\tau)d\tau\right\|\leqq C_n, \qquad nr<t\leqq(n+1)r.$$

Theorem 2. *If y is a Hölder continuous function in $[-r, 0]$ with values in the space $D(A)$ endowed with the graph norm of A and f is a Hölder continuous function in $[0, T]$ with values in X, then*

$$u(t)=W(t)y(0)+\int_{-r}^0 U_t(s)y(s)ds+\int_0^t W(t-s)f(s)ds,$$

where

$$U_t(s)=W(t-s-r)A_1+\int_{-r}^s W(t-s+\tau)a(\tau)A_2d\tau,$$

is a unique solution of (1) *in* $[0, T]$ *satisfying the initial condition*

$$u(s)=y(s), \qquad -r \leqq s \leqq 0.$$

Outline of proof. We essentially follow the method of J. Prüss [10]. In $[0, r]$ putting

$$V(t)=A(W(t)-T(t)) \tag{6}$$

we are led to the integral equation for $V(t)$:

$$V(t)=V_0(t)+\int_0^t A \int_\tau^t T(t-s)a(\tau-s)ds A_2 A^{-1} V(\tau)d\tau, \tag{7}$$

where

$$V_0(t)=\lim_{\varepsilon\to+0}\int_\varepsilon^t A \int_\tau^t T(t-s)a(\tau-s)ds A_2 T(\tau)d\tau.$$

It is not difficult to verify that $V_0(t)$ is strongly continuous in $[0, r]$. Hence, the equation (7) can be solved by successive approximation, and $W(t)$ defined by (6) is the desired fundamental solution in $[0, r]$.

Suppose that the existence of the fundamental solution $W(t)$ was established in $[0, nr]$. We construct $W(t)$ in $[nr, (n+1)r]$ as follows. We put

$$V(t)=A\Big(W(t)-\int_{nr}^t T(t-s)A_1 W(s-r)ds\Big).$$

Then, the integral equation to be satisfied by $V(t)$ is

$$V(t)=V_0(t)+\int_{nr}^t A \int_\tau^t T(t-s)a(\tau-s)ds A_2 A^{-1} V(\tau)d\tau,$$

where

$$\begin{aligned} V_0(t)=&AT(t)+A\int_r^{nr} T(t-s)A_1 W(s-r)ds \\ &+\int_0^{t-r} A \int_\tau^{\tau+r} T(t-s)a(\tau-s)ds A_2 W(\tau)d\tau \\ &+\int_{t-r}^{nr} A \int_\tau^t T(t-s)a(\tau-s)ds A_2 W(\tau)d\tau \\ &+\int_{nr}^t A \int_\tau^t T(t-s)a(\tau-s)ds A_2 \int_{nr}^\tau T(\tau-s)A_1 W(s-r)ds d\tau. \end{aligned}$$

It can be also shown without difficulty that $V_0(t)$ is strongly continuous in $[nr, (n+1)r]$. Hence, $W(t)$ can be constructed also in $[nr, (n+1)r]$. In the proof we need the Hölder continuity of $V(t)$:

$$\|V(t')-V(t)\| \leqq C_{n,\alpha}(t'-t)^\alpha (t-nr)^{-\alpha}, \quad nr \leqq t < t' \leqq (n+1)r, \quad 0<\alpha<\rho.$$

In a forthcoming paper we shall extend some of the results on the control theory which is being developed by S. Nakagiri and M. Yamamoto ([5]–[9]) to the case where operators in delay terms are unbounded for an equation in which A is the operator associated with a strongly elliptic sesquilinear form in a Hilbert space.

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