38. On the Complex C_Λ Attached to a Certain Class of Lagrangian Set

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0. Introduction. On a real manifold X, we prove that there exist microlocally simple sheaves along some kind of Laglangian set $\Lambda \subset T^*X$, and that such sheaves are unique up to shifts. This has been shown by M. Kashiwara and P. Schapira when Λ is smooth, and used in the course of the study of quantized contact transformations ([2], [3]). In this note we treat some cases where Λ is not smooth as well. As an application we can give a microlocal definition of the complex $C_{\Omega|X}$, which is introduced by P. Schapira [5] for the microlocal study of boundary value problems.

1. Let X be a C^2 manifold, and $\pi: T^*X \to X$ be the cotangent bundle to X. For $F \in Ob(D^+(X))$, the microsupport $SS(F) \subset T^*X$ is defined by Kashiwara-Schapira [1], [3]. For a point $p \in T^*X$, we denote by $D^+(X; p)$ the localization of the category $D^+(X)$ by the null system

$$\mathcal{E}(p) = \{F \in Ob(D^+(X)); SS(F) \not\ni p\}$$

(cf. [2], [3]).

Let Y be a closed submanifold of X, and Ω be an open subset of Y. We take a point $p \in T_Y^*X$, and assume, at $x = \pi(p) \in Y$,

$$N_x^*(\Omega) \neq T_x^*Y. \tag{1.1}$$

$N^*(\Omega)$ denotes the conormal cone of Ω in Y. We denote by ρ and $\tilde{\omega}$ the natural associated maps from $Y \times_X T^*X$ to T^*Y and T^*X, respectively.

Proposition 1.1. *Let $p \in T_Y^*X$ and let $F \in Ob(D^+(X))$. Assume* (1.1) *and $SS(F) \subset \tilde{\omega}\rho^{-1}(N^*(\Omega)^a \times_Y \bar{\Omega})$ in a neighbourhood of p. Then there exists a complex $M^\cdot$ of $\mathbf{Z}$-modules such that F is isomorphic to $M^\cdot_\Omega = M^\cdot \otimes_{\mathbf{Z}} \mathbf{Z}_\Omega$ in $D^+(X; p)$.*

Proof. Since $SS(F) \subset \pi^{-1}(Y)$, it follows from Proposition 6.2.1 of [3] that there exists $G \in Ob(D^+(Y))$ such that F is isomorphic to $\mathbf{R}j_*G$ in $D^+(X; p)$ (j denotes the embedding of Y into X). By Proposition 4.1.1 of [3] our assumption implies that $SS(G) \subset T^*Y$ is contained in $N^*(\Omega)^a \times_Y \bar{\Omega}$ in a neighbourhood of $x = \pi(p) \in T_Y^*Y$. Thus we have, taking account of (1.1),

i) $\mathrm{supp}(G) \subset \bar{\Omega}$,

ii) $SS(G) \cap N^*(\Omega) \subset T_Y^*Y$.

By Corollary 4.3.3 of [3], at any point $x' \notin \Omega$,

$$G_{x'} = \mathbf{R}\Gamma_{\bar{\Omega}}(G)_{x'} = 0.$$

This implies the natural morphism $G_\Omega \to G$ is an isomorphism in $D^+(Y; x)$.

Since $SS(G|_\Omega)\subset T^*_\Omega\Omega$ we can apply Proposition 4.1.2 of [3] to $G|_\Omega\in Ob(D^+(\Omega))$, and it remains to remark that by (1.1) there exists a fundamental system of neighbourhoods $\{\mathcal{U}\}$ of x such that $\mathcal{U}\cap\Omega$ is contractible. Q.E.D.

Remark. The Lagrangian set $\Lambda=SS(Z_\Omega)\subset T^*X$ is contained in $\tilde{\omega}\rho^{-1}(N^*(\Omega)\underset{Y}{\times}\bar{\Omega})$. Hence we can replace $\tilde{\omega}\rho^{-1}(N^*(\Omega)\underset{Y}{\times}\bar{\Omega})$ by Λ in Proposition 1.1.

From this proposition we get immediately the following corollaries.

Corollary 1.2. *Assume moreover F is simple with shift* $(1/2)\operatorname{codim} Y$ *on $\Omega\underset{Y}{\times}T^*_YX$, then F is isomorphic to Z_Ω in $D^+(X;p)$.*

Corollary 1.3. *Let F, G be two objects of $D^+(X)$. Assume that $SS(F)$ and $SS(G)$ are contained in $\tilde{\omega}\rho^{-1}(N^*(\Omega)^a\underset{Y}{\times}\bar{\Omega})$ in a neighbourhood of p. If F and G are isomorphic in $D^+(X;\Omega\underset{Y}{\times}T^*_YX)$, then they are isomorphic in $D^+(X;p)$.*

At the end we mention the case of closed subsets. Let X, Y be as above, and A be a closed subset of Y such that $N^*_x(A)\neq T^*_xY$.

Proposition 1.4. *Let $p\in T^*_YX$ and let $F\in Ob(D^+(X))$. Assume $SS(F)\subset\tilde{\omega}\rho^{-1}(N^*A\underset{Y}{\times}A)$ in a neighbourhood of p. Then there exists a complex $M^\cdot$ of Z-modules such that F is isomorphic to $M^\cdot_A$ in $D^+(X;p)$.*

The proof is similar to that of Proposition 1.1, but in this case we use the natural morphism $G\to \boldsymbol{R}\Gamma_\Omega(G)$ instead of $G_\Omega\to G$ ($\Omega=\operatorname{Int}(A)$).

2. We assume now X complex analytic, and denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X. Let Λ_0 be an $\boldsymbol{R}$-lagrangian and I-symplectic conic C^ω submanifold of T^*X (cf. [4]), and Λ_1 be an open conic subset of Λ_0 with C^ω boundary. We define $\Lambda\subset T^*X$ as the union of $\bar{\Lambda}_1$ and the half component of the union of the complex bicharacteristic leaves of $\partial\Lambda_1$ associated to Λ_1.

Theorem 2.1. *Let $p\in\Lambda\cap\Lambda_0$. Then there exists a unique $F_\Lambda\in Ob(D^+(X;p))$ such that $SS(F_\Lambda)\subset\Lambda$ and F_Λ is simple on Λ_1 with shift* $(1/2)s-(1/2)n$ $(n=\dim_C X)$ *where*

$$s=\frac{1}{2}\tau(T_p\Lambda_0, iT_p\Lambda_0, T_p(\pi^{-1}\pi(p))),$$

and τ denotes the Maslov index with respect to the real symplectic structure of T_pT^*X (cf. [2]).

Proof. By Lemma 1.2 of [4] we can find a complex homogeneous symplectic transformation $\varphi: T^*X\to T^*C^n$ which interchanges (Λ_0, Λ) and $(T^*_{R^n}C^n, T^*_\Omega C^n\,(=SS(Z_\Omega)))$ for some open subset $\Omega\subset R^n$ with C^ω boundary.

Let φ_K be a quantized contact transformation over φ (cf. [2], [3]) for $K\in Ob(D^b_{R-c}(X\times C^n))$, simple with shift n along the lagrangian variety associated to φ. Then $F_\Lambda=\varphi_K^{-1}(Z_\Omega)[-n]$ satisfies the conditions. Such F_Λ is unique in $D^+(X;p)$ by Corollary 1.3. Q.E.D.

Let X' be a copy of X.

Proposition 2.2. *Let $p \in \Lambda \cap \Lambda_0$ and let $\varphi : (T^*X, p) \to (T^*X', p')$ be a germ of a complex homogeneous symplectic transformation at p interchanging Λ and Λ' ($\Lambda' \subset T^*X'$: a lagrangian subset). Then there exists a natural isomorphism*:

$$\varphi_K F_\Lambda \cong F_{\Lambda'} \quad \text{in } D^+(X'; p'), \tag{2.1}$$

where $\varphi_K : D^+(X; p) \to D^+(X'; p')$ is a quantized contact transformation over φ for a simple sheaf K with shift n.

This proposition follows easily from the uniqueness of the complex F_Λ.

Let Λ be a lagrangian subanalytic set given as above. We define the complex C_Λ on Λ by:

$$C_\Lambda = \mu\, \mathrm{hom}\, (F_\Lambda, \mathcal{O}_X). \tag{2.2}$$

It is directly deduced from Proposition 2.2 and Theorem 4.1 of [2] that C_Λ is defined independently of the choice of the complex symplectic coordinate system of T^*X. In other words, in the situation of Proposition 2.2, we have a natural isomorphism (i.e., quantized contact transformation) at $p' = \varphi(p)$:

$$\varphi_* C_\Lambda \xrightarrow{\sim} C_{\Lambda'}. \tag{2.3}$$

In particular, in the situation of the proof of Theorem 2.1, we have:

$$\varphi_* C_\Lambda \cong C_{\Omega|X}. \tag{2.4}$$

We note that $C_{\Omega|X}$ is defined for any open subset $\Omega \subset \boldsymbol{R}^n$ in [5].

Remark. In case $\Lambda_0 = \Lambda_1 = \Lambda$, C_Λ is nothing but the sheaf of Sato's microfunctions ([8]).

Let X be a complex neighbourhood of a real analytic manifold M, and Ω be an open subset of M satisfying the cone condition (1.1):

$$N_x^*(\Omega) \neq T_x^* M.$$

We denote by (M', X') a copy of (M, X).

Corollary 2.3. *Let $p \in T_M^* X_x$ and let $\varphi : (T^*X, p) \to (T^*X', p')$ be a germ of a complex homogeneous symplectic transformation at p interchanging $T_\Omega^* X$ and $T_{\Omega'}^* X'$ ($\Omega' \subset M'$: an open subset). Then there exists a natural isomorphism at p'*:

$$\varphi_* C_{\Omega|X} \xrightarrow{\sim} C_{\Omega'|X'}. \tag{2.5}$$

Remark. The similar result on the complex $C_{M_+|X}$ ($M_+ \subset M$: a closed subset of M) can be deduced from Proposition 1.4 (cf. [5] and Kataoka [7] for the definition of $C_{M_+|X}$). That for the case where M_+ has C^ω boundary has been obtained by Kataoka [7].

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