36. *Linear Extensions and Order Polynomials of Finite Partially Ordered Sets*

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Any partially ordered set (*poset* for short) to be considered is finite. The cardinality of a finite set X is denoted by $\sharp(X)$. Let N be the set of non-negative integers and Z the set of integers.

Introduction. Let P be a poset with elements $x_1, x_2, \cdots, x_p$ labeled so that if $x_i < x_j$ in P then $i<j$ in Z. Given an integer i, $0 \leq i < p$, write $w_i = w_i(P)$ for the number of permutations $\pi = \begin{pmatrix} 1 & 2 & \cdots p \\ a_1 & a_2 & \cdots a_p \end{pmatrix}$ such that (a) if $x_{a_r} < x_{a_s}$ in P, then $r<s$ (i.e., π is a *linear extension* of P) and (b) $\sharp\{r\,;\, a_r > a_{r+1}\}$, the number of *descents* of π, is equal to i. Let $s = \max\{i\,;\, w_i \neq 0\}$. We say that the vector $w(P) = (w_0, w_1, \cdots, w_s)$ is the *w-vector* of P.

On the other hand, for any $n \in N$ we write $\Omega(P, n)$ for the number of maps σ from P to N such that (a) if $x_i < x_j$ in P then $\sigma(x_i) \geq \sigma(x_j)$ and (b) $\max\{\sigma(x_i)\,;\, 1 \leq i \leq p\} \leq n$. It is known that $\Omega(P, n)$ is a polynomial, called the *order polynomial* of P, for n sufficiently large and the degree of this polynomial is p. A fundamental relation between $\Omega(P, n)$ and $w(P)$ is the equality

$$(1-\lambda)^{p+1} \sum_{n=0}^{\infty} \Omega(P, n)\lambda^n = w_0 + w_1\lambda + \cdots + w_s\lambda^s.$$

Consult [5, Chapter 4, Section 5] for further information.

A big open question in enumerative combinatorics is to characterize the w-vectors of posets. Recently, Stanley obtained the linear inequalities

$$w_0 + w_1 + \cdots + w_i \leq w_s + w_{s-1} + \cdots + w_{s-i}, \qquad 0 \leq i \leq [s/2]$$

for the w-vector $w(P) = (w_0, w_1, \cdots, w_s)$ of an arbitrary poset P. We can go on to ask, what more can be said about the w-vector of a poset? In what follows, after summarizing notation and terminology, we give new inequalities for the w-vector of a poset which satisfies a certain chain condition. Systematic study of w-vectors, including detailed proofs of our results, will be found in [2].

Notation and terminology. A *chain* is a poset in which any two elements are comparable. The *length* of a chain C is defined by $\ell(C) := \sharp(C) - 1$. The *rank* of a poset P, denoted by rank(P), is the supremum of lengths of chains contained in P. If $\alpha \leq \beta$ in P, then we write $\ell(\alpha, \beta)$ for the rank of the subposet $P_\alpha^\beta := \{x \in P\,;\, \alpha \leq x \leq \beta\}$ of P. A poset P is called *pure* if every maximal chain of P has the same length. We say that P satisfies the $\delta^{(n)}$-*chain condition*, $n \in N$, if (a) for any $\xi \in P$, the subposet $P_\xi := \{y \in P\,;\, y \geq \xi\}$

of P is pure and (b) $\mathrm{rank}(P)-\min\{\ell(C); C$ is a maximal chain of $P\}=n$. Thus P satisfies the $\delta^{(0)}$-chain condition if and only if P is pure.

Given a poset P, we write $P^{\wedge}$ for the poset obtained by adjoining a *new* pair of elements, $0^{\wedge}$ and $1^{\wedge}$, to P such that $0^{\wedge}<x<1^{\wedge}$ for any $x\in P$. A sequence $\mathscr{A}=(\alpha_0, \beta_0, \alpha_1, \beta_1, \cdots, \alpha_t, \beta_t)$, which consists of elements of $P^{\wedge}$, is called *rhythmical* if (a) $\alpha_0=0^{\wedge}$, $\beta_t=1^{\wedge}$, (b) $\alpha_i<\beta_i$ for and i, $0\leq i\leq t$, (c) $\alpha_{i+1}<\beta_i$ for any i, $0\leq i<t$ and (d) $\alpha_{i+2}\not<\beta_i$ for any i, $0\leq i\leq t-2$. Let $\ell(\mathscr{A}):=\sum_{0\leq i\leq t}\ell(\alpha_i, \beta_i)-\sum_{0\leq i\leq t-1}\ell(\alpha_{i+1}, \beta_i)$. We say that P satisfies the *Δ-chain condition* if $\ell(\mathscr{A})\leq\mathrm{rank}(P^{\wedge})$ for any rhythmical sequence $\mathscr{A}$ of $P^{\wedge}$. We easily see that, for any $n\in N$, the $\delta^{(n)}$-chain condition implies the Δ-chain condition.

Results. First, we state non-linear inequalities for the w-vector of a poset which satisfies the Δ-chain condition.

Theorem. *Assume that a poset P with $w(P)=(w_0, w_1, \cdots, w_s)$ satisfies the Δ-chain condition. If i and j are non-negative integers with $i+j\leq s$, then $w_i\leq w_j w_{i+j}$.*

Secondly, if a poset P satisfies the $\delta^{(n)}$-chain condition, then certain linear inequalities hold for the w-vector $w(P)$, that is to say,

Theorem. *Let $w(P)=(w_0, w_1, \cdots, w_s)$ be the w-vector of a poset P satisfying the $\delta^{(n)}$-chain condition. Then we have the inequality*

$$w_s+w_{s-1}+\cdots+w_{s-i}\leq w_0+w_1+\cdots+w_i+\cdots+w_{i+n}$$

for any i, $0\leq i\leq[(s-n)/2]$.

Our technique [2], which originated in [1], is heavily based on commutative algebra, especially the theory of canonical modules [4] of invariant subrings of tori [3].

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